Kaluza-Klein Gravity

J. M. Overduin

Department of Physics and Astronomy

University of Victoria, British Columbia, Canada

and

P. S. Wesson

Department of Physics

University of Waterloo, Ontario, Canada

Original Informative from Internet[1]

[1] **Documento:** "... gaia/en/educacional/science/dimensionc/kaluza_klein_gravity.pdf".
[2] **Internet:** "http://arxiv.org/pdf/gr-qc/9805018.pdf".

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J. M. Overduin

Department of Physics and Astronomy, University of Victoria, P.O. Box 3055,

Victoria, British Columbia, Canada, V8W 3P6

and

P. S. Wesson

Department of Physics, University of Waterloo, Ontario, Canada N2L 3G1

and

Gravity Probe-B, Hansen Physics Laboratories, Stanford University, Stanford, California, U.S.A. 94305

arXiv:gr-qc/9805018 v1 7 May 1998

Preprint submitted to Elsevier Preprint *20 April 2001*

0.1 Abstract

We review higher-dimensional unified theories from the general relativity, rather than the particle physics side. Three distinct approaches to the subject are identified and contrasted: compactified, projective and noncompactified. We discuss the cosmological and astrophysical implications of extra dimensions, and conclude that none of the three approaches can be ruled out on observational grounds at the present time.

1 Introduction

Kaluza's [1] achievement was to show that five-dimensional general relativity contains both Einstein's four-dimensional theory of gravity and Maxwell's theory of electromagnetism. He however imposed a somewhat artificial restriction (the cylinder condition) on the coordinates, essentially barring the fifth one a *priori* from making a direct appearance in the laws of physics. Klein's [2] contribution was to make this restriction less artificial by suggesting a plausible *physical* basis for it in compactification of the fifth dimension. This idea was enthusiastically received by unified-field theorists, and when the time came to include the strong and weak forces by extending Kaluza's mechanism to higher dimensions, it was assumed that these too would be compact. This line of thinking has led through eleven-dimensional supergravity theories in the 1980s to the current favorite contenders for a possible "theory of everything," ten-dimensional superstrings.

We review the field of Kaluza-Klein gravity, concentrating on the general relativity, rather than particle physics side of the subject. (For the latter there are already many excellent books [3]-[8] and review articles [9]-[14] available.) We also aim to *re-examine* the field to some extent, as it seems to us that the cart of compactification has in some ways gotten ahead of the horse of unification. Kaluza unified not only gravity and electromagnetism, but also *matter and geometry*, for the photon appeared in four dimensions as a manifestation of *empty* five-dimensional spacetime. Modern Kaluza-Klein theories, by contrast, routinely require the addition of explicit "higher-dimensional matter" fields in order to achieve successful compactification (among other things). Are they necessary? Yes, if extra coordinates must be real, lengthlike and compact. There are, however, higher-dimensional unified field theories which require none of these things: *projective* theories [15]-[18], in which extra coordinates are not physically real; and *noncompactified* theories [19]-[26], in which they are not necessarily lengthlike or compact. These theories receive special attention in our report.

We begin in §2 with a historical overview of higher-dimensional theories of gravity. In §3 we review Kaluza's original mechanism, emphasizing what to us are its three principal features. The three main approaches to higher-dimensional unification *since* Kaluza - compactified, projective and

noncompactified - are reviewed in §4, §5 and §6 respectively. We note that each one modifies or sacrifices at least one of the key features of Kaluza's theory, and discuss the implications. The all-important question of experimental constraints is addressed in §7 and §8, which deal respectively with cosmological and astrophysical effects of extra dimensions. None of the three above-mentioned approaches can be ruled out on observational grounds at the present time. Conclusions and prospects for further work are summarized in §9.

2 Historical Overview

2.1 Higher Dimensions

The world of everyday experience is three-dimensional. But why should this be so? The question goes back at least to Kepler [27], who speculated that the threefold nature of the Holy Trinity might be responsible. More recent arguments have involved the stability of planetary orbits and atomic ground states, the use of wave propagation for information transmission, the fundamental constants of nature, and the anthropic principle [28], as well as wormhole effects [29], the cosmological constant [30], certain "geometry-free" considerations [31], string theories [32], and nucleation probabilities in quantum cosmology [33]. All these lines of reasoning converge on the same conclusion: that, in agreement with common intuition, space is composed of three macroscopic spatial dimensions x^1, x^2 and x^3.

Nevertheless, the temptation to tinker with the dimensionality of nature has proved irresistible to physicists over the years. The main reason for this is that phenomena which require very different explanations in three-dimensional space can often be shown to be manifestations of simpler theories in higher-dimensional manifolds. But how can this idea be reconciled with the observed three-dimensionality of space? If there are additional coordinates, why does physics appear to be independent of them?

It is useful to keep in mind that the new coordinates need not necessarily be lengthlike (in the sense of being measured in meters, say), or even spacelike (in regard to their metric signature). A concrete example which violates both of these expectations was introduced in 1909 by Minkowski [34], who showed that the successes of Maxwell's unified electromagnetic theory and Einstein's special relativity could be understood geometrically if time, along with space, were considered part of a four-dimensional spacetime manifold via $x^0 = ict$. Many of the abovementioned arguments against more than three dimensions were circumvented by the fact that the fourth coordinate did not mark *distance*. And the reason that physics had appeared three-dimensional for so long was because of the large size of the dimension-transposing parameter c, which meant that the effects of "mixing" space and time coordinates (ie., length contraction, time dilation) appeared only at very high speeds.

2.2 Kaluza-Klein Theory

Inspired by the close ties between Minkowski's four-dimensional spacetime and Maxwell's unification of electricity and magnetism, Nordström [35] in 1914 and (independently) Kaluza [1] in 1921 were the first to try unifying *gravity* with electromagnetism in a theory of *five* dimensions (x^0 through x^4). Both men then faced the question: why had no fifth dimension been observed in nature? In Minkowski's time, there had already been experimental phenomena (namely, electromagnetic ones) whose invariance with respect to Lorentz transformations could be interpreted as four-dimensional coordinate invariance. No such observations pointed to a fifth dimension. Nordström and Kaluza therefore avoided the question and simply demanded that all derivatives with respect to x^4 vanish. In other words, physics was to take place - for as-yet unknown reasons - on a four-dimensional

hypersurface in a five-dimensional universe (Kaluza's "cylinder condition").

With this assumption, each was successful in obtaining the field equations of both electromagnetism and gravity from a single five-dimensional theory. Nordström, working as he was before general relativity, assumed a scalar gravitational potential; while Kaluza used Einstein's tensor potential. Specifically, Kaluza demonstrated that general relativity, when interpreted as a five-dimensional theory in vacuum (ie., ${}^5G_{AB} = 0$, with A, B running over 0, 1, 2, 3, 4), *contained* four-dimensional general relativity in the presence of an electromagnetic field (ie., ${}^4G_{\alpha\beta} = {}^4 T^{EM}_{\alpha\beta}$, with α, β running over 0, 1, 2, 3), together with Maxwell's laws of electromagnetism. (There was also a Klein-Gordon equation for a massless scalar field, but this was not appreciated - and was in fact suppressed - by Kaluza at the time.) All subsequent attempts at higher-dimensional unification spring from this remarkable result.

Various modifications of Kaluza's five-dimensional scheme, including Klein's idea [2], [36] of compactifying the extra dimension (which we will discuss in a moment) were suggested by Einstein, Jordan, Bergmann, and a few others [37]-[43] over the years, but it was not extended to *more* than five dimensions until theories of the strong and weak nuclear interactions were developed. The obvious question was whether these new forces could be unified with gravity and electromagnetism by the same method.

The key to achieving this lay in the concept of gauge invariance, which was coming to be recognized as underlying all the interactions of physics. Electrodynamics, for example, could be "derived" by imposing local $U(1)$ gauge-invariance on a free-particle Lagrangian. From the gauge-invariant point of view, Kaluza's feat in extracting electromagnetism from five-dimensional gravity was no longer so surprising: it worked, in effect, because $U(1)$ *gauge-invariance* had been "added onto" Einstein's equations in the guise of *invariance with respect to coordinate transformations* along the fifth dimension. In other words, gauge symmetry had been "explained" as a geometric symmetry of spacetime. The electromagnetic field then appeared as a vector "gauge field" in four dimensions. It was natural - though not simple - to extend this insight to groups with more complicated symmetry. De Witt [44] in 1963 was the first to suggest incorporating the non-Abelian $SU(2)$ gauge group of Yang and Mills into a Kaluza-Klein theory of (4 + d) dimensions. A minimum of three extra dimensions were required. This problem was picked up by others [45]-[47] and solved completely by the time of Cho & Freund [48], [49] in 1975.

2.3 Approaches to Higher-Dimensional Unification

We emphasize here three key features of all the models discussed so far:

(*i*) They embody Einstein's vision [50]-[52] of nature as *pure geometry*. (This idea can be traced in nonmathematical form at least to Clifford in 1876 [53], and there are hints of it as far back as the Indian Vedas, according to Wheeler and others [54].) The electromagnetic and Yang-Mills fields, as well as the gravitational field, are completely contained in the higher-dimensional Einstein tensor ${}^{(4+d)}G_{AB}$; that is, in the metric and its derivatives. No explicit energy-momentum tensor ${}^{(4+d)}T_{AB}$ is needed.

(*ii*) They are *minimal* extensions of general relativity in the sense that there is no modification to the mathematical structure of Einstein's theory. The only change is that tensor indices run over 0 to (3 + d) instead of 0 to 3.

(*iii*) They are a *priori* cylindrical. No mechanism is suggested to explain why physics depends on the first four coordinates, but not on the extra ones.

The first two of these are agreeable from the point of view of elegance and simplicity. The third, however, appears contrived to modern eyes. In the effort to repair this defect, higher-dimensional unified theory has evolved in three more or less independent directions since the time of Kaluza.

Each one sacrifices or modifies one of the features (*i*) to (*iii*) above.

Firstly, it has been proposed that extra dimensions do not appear in physics because they are *compactified* and unobservable on experimentally accessible energy scales. This approach has been successful in many ways, and is the dominant paradigm in higher-dimensional unification (recent reviews include many excellent books [3]-[8] and articles [9]-[14]). If one wants to unify more than just gravity and electromagnetism in this way, however, it seems that one has in practice to abandon Einstein's goal of geometrizing physics, at least in the sense of (*i*) above.

A second way to sweep the extra dimensions out of sight is to regard them as mathematical artifacts of a more complicated underlying theory, sacrificing (*ii*) above. This can be done, for example, if one replaces the classical (affine) geometry underlying Einstein's general relativity with *projective* geometry (see for reviews [15]-[18]). "Extra dimensions" then become visual aids which may or may not help us understand the underlying mathematics of nature, but which do not correspond to physical coordinates.

The third approach to the problem of explaining exact cylindricity is to consider the possibility that it *may not necessarily be* exact, relaxing (*iii*) above. That is, one takes the new coordinates at face value, allowing physics to depend on them in principle [19]-[26]. This dependence presumably appears in regimes that have not yet been well-probed by experiment - much as the relevance of Minkowski's fourth dimension to mechanics was not apparent at non-relativistic speeds. When dependence on the extra dimensions is included, one finds that the five-dimensional Einstein equations ${}^5R_{AB} = 0$ contain the four-dimensional ones ${}^4G_{\alpha\beta} = {}^4T_{\alpha\beta}$ with a *general* energy-momentum tensor ${}^4T_{\alpha\beta}$ instead of just the electromagnetic one ${}^4T^{EM}_{\alpha\beta}$.

2.4 The Compactified Approach

Klein showed in 1926 [2], [36] that Kaluza's cylinder condition would arise naturally if the fifth coordinate had (1) a circular topology, in which case physical fields would depend on it only periodically, and could be Fourier-expanded; and (2) a small enough ("compactified") scale, in which case the energies of all Fourier modes above the ground state could be made so high as to be unobservable (except - as we now add - possibly in the very early universe). Physics would thus be effectively independent of Kaluza's fifth dimension, as desired. As a bonus, it seemed early on that the expansion of the electromagnetic field into Fourier modes could in principle explain the quantization of electric charge. (This aspect of the theory has had to be abandoned, however, as the charge-to-mass ratio of the higher modes did not match that of any known particles. Nowadays elementary charges are identified with the ground state Fourier modes only, and their small mass is attributed to spontaneous symmetry-breaking.)

The scheme was not perfect; one still needed to explain *why* extra dimensions differed so markedly in topology and scale from the familiar spacetime ones. Their size in particular had to be extremely small (below the attometer ($1am = 10^{-18}m$) scale, according to current experiment [55]). There was also the question of how to interpret a new scalar field which appeared in the theory. These difficulties have, however, proved manageable. Scalar fields are not as threatening as they once appeared; one now just assumes that they are too massive to have been observed. And an entire industry has grown up around the study of compactification mechanisms and the topology of compact spaces.

In fact, Klein's strategy of compactifying extra dimensions has come to dominate higher-dimensional unified physics, leading in recent years to new fields like eleven-dimensional supergravity and ten-dimensional superstring theory. We will survey these developments in this section, and make contact with many of them throughout this report, but it is not our purpose to review them exhaustively. For this the reader is directed to the books [3]-[8], and review articles [9]-[14] mentioned already. Our goal here is to take a broad view, comparing and contrasting the various approaches to higher-

dimensional gravity, and focusing in particular on those which have received less critical attention in the literature. A semantic note: while the term "Kaluza-Klein theory" ought, strictly speaking, to apply only to models which assume both cylindricity and compactified dimensions, we follow popular usage and apply the term to any higher-dimensional unified theory of gravity in which the extra dimensions are regarded as real, whether compactified or not. When distinguishing between these two, we will refer in the latter case to "noncompactified Kaluza-Klein theories," though this is to some extent a contradiction in terms.

2.5 Compactification Mechanisms

A difficulty with compactification is that one cannot impose it indiscriminately on whichever dimensions one likes - the combination of macroscopic four-dimensional spacetime plus the compactified extra-dimensional space must be a solution of the higher-dimensional Einstein field equations. In particular, one should be able to recover a "ground state" solution consisting of four-dimensional Minkowski space plus a d-dimensional compact manifold. Although this is straightforward when $d = 1$ (Klein's case), the same thing is not true in higher-dimensional theories like that of Cho & Freund, where the compact spaces are in general curved [7], [56]-[59]. The consequences of ignoring this inconsistency in "Kaluza-Klein ansatz" have been emphasized by Duff *et al.* [11], [12], [14], [60]. This and related problems have even led Cho [61]-[64] to call for the abandonment of Klein's "zero modes approximation" as a means of dimensional reduction.

In general, however, spacetime can still be coaxed into compactifying in the desired manner - at the cost of altering the higher-dimensional vacuum Einstein equations, either by incorporating torsion [65]-[68], adding higher-derivative terms (eg., R^2) onto the Einstein action [69], or - last but not least - adding an explicit higher-dimensional energy-momentum tensor to the theory. If chosen judiciously, this last will induce "spontaneous compactification" of the extra dimensions, as first demonstrated by Cremmer & Scherk [70], [71]. This approach, though, sacrifices Einstein and Kaluza's dream [50]-[54] of a purely *geometrical* unified theory of nature. Rather than explaining the "base wood" of four-dimensional matter and forces as manifestations of the "pure marble" of geometry in higher dimensions, one has essentially been driven to invent *new* kinds of wood. Weinberg [72] has likened this situation to the fable of "stone soup," in which a miraculous stew, allegedly made out of rocks, turns out on deeper investigation to be made from rocks plus various kinds of vegetables, meat and spices.

In spite of this aesthetic drawback, however, the idea of spontaneous compactification gained rapid acceptance [73]-[78] and has become the standard way to reconcile extra dimensions with the observed four-dimensionality of spacetime in Kaluza-Klein theory (see Bailin & Love [13] for a review). An important variation is that of Candelas & Weinberg [79], [80], who showed that the quantum Casimir energy of massless higher-dimensional fields, when combined with a higher-dimensional cosmological constant, can also compactify the extra dimensions in a satisfactory way. Unfortunately, some $10^4 - 10^5$ matter fields are required.

2.6 D = 11 Supergravity

One way to make the addition "by hand" of extra matter fields more natural was to make the theory *supersymmetric* (ie., to match up every boson with an as-yet undetected fermionic "superpartner" and vice versa). The reason for this is that the (compactified) Kaluza-Klein programme of "explaining" gauge symmetries as (restricted) higher-dimensional spacetime symmetries can only give rise to four-dimensional gauge *bosons*. If the theory is to include *fermionic* fields, as required by supersymmetry, then these fields at least must be put in by hand. (This limitation may not necessarily apply to noncompactified Kaluza-Klein theories, in which the modest dependence on extra coordinates -

subject to experimental constraints - gives the Einstein equations a rich enough structure that matter of a very *general* kind can be "induced" in the four-dimensional universe by pure geometry in higher dimensions. In five dimensions, for example, one can obtain not only photons, the gauge bosons of electromagnetism, but also dustlike, vacuum, or "stiff" matter.)

Supersymmetric gravity ("supergravity") began life as a four-dimensional theory in 1976 [81], [82], but quickly made the jump to higher dimensions ("Kaluza-Klein supergravity"). It was particularly successful in $D = 11$, for three principal reasons. First, Nahm [83] showed that eleven was the *maximum* number of dimensions consistent with a single graviton (and an upper limit of two on particle spin). This was followed by Witten's proof [84] that eleven was also the *minimum* number of dimensions required for a Kaluza-Klein theory to unify all the forces in the standard model of particle physics (ie., to contain the gauge groups of the strong ($SU(3)$) and electroweak ($SU(2) \times U(1)$) interactions). The combination of supersymmetry with Kaluza-Klein theory thus appeared to *uniquely fix* the dimensionality of spacetime. Secondly, whereas in lower dimensions one had to choose between several possible configurations for the extra matter fields, Cremmer, Julia & Scherk [85] demonstrated in 1978 that in $D = 11$ exactly *one* choice was consistent with the requirements of supersymmetry (in particular, that there be equal numbers of Bose and Fermi degrees of freedom). In other words, while a higher-dimensional energy-momentum tensor was still required, its form at least appeared less contrived. Finally, Freund & Rubin [86] showed in 1980 that compactification of the $D = 11$ model could occur in only two ways: to seven or four compact dimensions, leaving four (or seven, respectively) macroscopic ones. Not only did eleven-dimensional spacetime appear to be uniquely favoured for unification, but it also split perfectly to produce the observed four-dimensional world. (The other possibility, of a macroscopic *seven*-dimensional world, could unfortunately not be ruled out, and in fact at least one such model was explicitly constructed as well [87].) Buoyed by these successes, eleven-dimensional supergravity appeared set by the mid 1980s as a leading candidate for the hoped-for "theory of everything" (see [8], [12], [88] for reviews, and [89] for an extensive collection of papers. A nontechnical introduction is given in [90].)

A number of blemishes, however - one aesthetic and three more practical - have dampened this initial enthusiasm. Firstly, the compact manifolds originally envisioned by Witten [84] (those containing the standard model) turned out not to generate quarks or leptons, and to be incompatible with supersymmetry [8], [13]. Their most successful replacements are the 7-sphere and the "squashed" 7-sphere [12], described respectively by the symmetry groups $SO(8)$ and $SO(5) \times SU(2)$. These groups, however, unfortunately do not contain the minimum symmetry requirements of the standard model ($SU(3) \times SU(2) \times U(1)$). This is commonly rectified by adding *more* matter fields, the "composite gauge fields" [91], to the eleven-dimensional Lagrangian. Secondly, it is very difficult to build chirality (necessary for a realistic fermion model) into an eleven-dimensional theory [84], [92]. A variety of remedies have been proposed for this, including the ubiquitous additional higher-dimensional gauge fields [73], [77], *noncompact* internal manifolds [93]-[99], and extensions of Riemannian geometry [100]-[102]. Thirdly, $D = 11$ supergravity theory is marred by a large cosmological constant in four dimensions, which is difficult to remove, even by fine-tuning [7], [8]. Finally, quantization of the theory leads inevitably to anomalies [89].

Some of these difficulties can be eased by descending to ten dimensions: chirality is easier to obtain [92], and many of the anomalies disappear [103]. However, the introduction of chiral fermions leads to *new* kinds of anomalies. And the primary benefit of the $D = 11$ theory - its uniqueness - is lost, since ten dimensions are not specially favoured, and the higher-energy theory does not break down naturally into four macroscopic and six compact dimensions. (One can still find solutions in which this happens, but there is no reason why they should be preferred.) In fact, most $D = 10$ supergravity models not only require ad hoc higher-dimensional matter fields to ensure proper compactification, but entirely *ignore* gauge fields arising from the Kaluza-Klein mechanism (ie., from symmetries of the compact manifold), so that *all* the gauge fields are effectively put into the theory by hand [8]. Kaluza's original aim of explaining forces in geometrical terms is thus abandoned completely.

2.7 D = 10 Superstring Theory

A breakthrough in solving the uniqueness and anomaly problems of $D = 10$ theory occured when Green & Schwarz [104] and Gross *et al.* [105] showed that there were two, and *only* two ten-dimensional supergravity models in which *all* anomalies could be made to vanish: those based on the groups $SO(32)$ and $E_8 \times E_8$ respectively. Once again, extra terms (known as Chapline-Manton terms) had to be added to the higher-dimensional Lagrangian [8]. This time, however, the addition was not completely arbitrary; the extra terms were those which would appear anyway if the theory were a low-energy approximation to certain kinds of super*string* theory.

The state of the art in compactified Kaluza-Klein theory, then, has shifted from supergravity theories to superstring theories, the main significance of the former now being as low-energy limits of the latter [89]. Superstrings (supersymmetric generalizations of strings) avoid the generic prediction of tachyons that plagued the first string theories [106], but retain their best features, especially the possibility of an anomaly-free path to quantum gravity [107]. In fact, their many virtues make them the current favorite contender for a "theory of everything" [108]. Connections have recently been made between certain superstring states and extreme black holes [14], and it has even been argued that superstrings can help resolve the long-standing black hole information paradox [109].

Something of a uniqueness problem has persisted for $D = 10$ superstrings in that the groups $SO(32)$ and $E_8 \times E_8$ admit five different string theories between them. But this difficulty has recently been addressed by Witten [376], who showed that it is possible to view these five theories as aspects of a single underlying theory, now known as M-theory (for "Membrane") [377]. The low-energy limit of this new theory, furthermore, turns out to be $D = 11$ supergravity! So it appears that the preferred dimensionality of spacetime in compactified Kaluza-Klein theory may be switching back to eleven.

Perhaps the biggest obstacle to a wider acceptance of these theories is the difficulty of extracting clear-cut physical predictions from them. String theory is "promising ...," one worker has said, "...and promising, and promising" [110]. M-theory, which (unlike superstring theory) is not perturbative, is even more opaque; Witten has suggested that the "M" might equally well stand for "Magic" or "Mystery" at present [378]. We will not consider these interesting developments further here, directing the reader instead to the superstring reviews in [8], [111], [112] (a nontechnical account may be found in [113]), or the recent reviews of M-theory in [378], [379].

2.8 The Projective Approach

Compactification of extra dimensions is not the only way to explain Kaluza's cylinder condition. Another, less well-known approach goes back to Veblen & Hoffmann [114] in 1931. These authors showed that the fifth dimension could be "absorbed into" ordinary four-dimensional spacetime if one replaced the *classical* (affine) tensors of general relativity with *projective* ones. Rather than being regarded as new coordinates, the extra dimensions were effectively demoted to visual aids. Because they were not physically real, there was no need to explain why they were not observed. The price for this resolution of the problem was that one had to alter the geometrical foundation of Einstein's theory. This idea received attention from Jordan, Pauli and several others over the years [39], [115]-[122]. Early versions of the theory ran afoul of experimental constraints on the Brans-Dicke parameter ω and had to be ruled out as untenable [15]. However the projective approach has been revived in at least two new formulations. That of Lessner [15] assigns the scalar field a purely *microscopic* meaning; this has interesting consequences for elementary particles [123]-[126]. The other, due to Schmutzer [16]-[18], endows the vacuum with a special kind of higher-dimensional matter, the "non-geometrizable substrate," thereby sacrificing Einstein's dream of reducing physics to geometry. This theory, however, does make a number of testable predictions [17], [127]-[129] which

are so far compatible with observation.

2.9 The Noncompactified Approach

An alternative to both the compactified and projective approaches is to take the extra dimensions at face value, without necessarily compactifying them, and assume that nature is only *approximately* independent of them - much as it was on Minkowski's fourth coordinate at nonrelativistic speeds. In other words, one avoids having to explain why cylindricity should be exact by *relaxing* it in principle. Of course, the question remains as to why nature should be so *nearly* cylindrical in practice. If the extra dimensions are lengthlike, then one might try to answer this by supposing that particles are trapped near a four-dimensional hypersurface by a potential well. Ideas of this kind have been around since at least 1962 [130]; for recent discussions see [131]-[134].

Confining potentials are not, however, an obvious improvement over compactification mechanisms in terms of economy of thought. An alternative is to take Minkowski's example more literally and entertain the idea that extra dimensions, like time, might not necessarily be lengthlike. In this case the explanation for the near-cylindricity of nature is to be found in the physical interpretation of the extra coordinates; ie., in the values of the dimension-transposing parameters (like c) needed to give them units of length. The first such proposal of which we are aware is the 1983 "space-time-mass" theory of Wesson [135], who suggested that a fifth dimension might be associated with *rest mass* via $x^4 = Gm/c^2$. The chief effect of this new coordinate on four-dimensional physics was that particle rest mass, usually assumed to be constant, varied with time. The variation was, however, small and quite consistent with experiment. This model has been studied in some detail, particularly with regard to its consequences for astrophysics and cosmology, by Wesson [19], [136]-[139] and others [140]-[148], [149]-[158], [159]-[163], and has been extended to *more* than five dimensions by Fukui [164], [165], with the constants $\hbar$ and e playing roles analogous to c and G.

Variable gravity theories are, of course, not new. What *is* new in the models just described - and what is important about noncompactified Kaluza-Klein theory in principle - is not so much the particular physical interpretation one attaches to the new coordinates, but the bare fact that *physics is allowed to depend on them at all.* It is clearly of interest to study the higher-dimensional Einstein equations with a *general* dependence on the extra coordinates; ie., without any preconceived notions as to their physical meaning. A pioneering effort in this direction was made in 1986 by Yoshimura [166], who however considered only the case where the d-dimensional part of the $(4 + d)$-space could depend on the new coordinates. The general theory, in which *any* part of the metric can depend on the fifth coordinate, has been explored recently by Wesson and others [19]-[26], and its implications for cosmology [167]-[170], [171]-[179] and astrophysics [180]-[185], [186]-[194] have become the focus of a growing research effort. As this branch of Kaluza-Klein theory has not yet been reviewed in a comprehensive manner, we propose to devote special attention to it in this report. Our intention is to compare and contrast this branch of the subject with other ones, however, so we will make frequent contact with the compactified and (to a lesser extent) projective theories.

3 The Kaluza Mechanism

Kaluza unified electromagnetism with gravity by applying Einstein's general theory of relativity to a *five*-, rather than four-dimensional spacetime manifold. In what follows, we consider generalizations of his procedure that may be new to some readers, so it will be advantageous to briefly review the mathematics and underlying assumptions here.

3.1 Matter from Geometry

The Einstein equations in five dimensions with *no five-dimensional energy-momentum tensor* are:

$$\hat{G}_{AB} = 0 \quad , \tag{1}$$

or, equivalently:

$$\hat{R}_{AB} = 0 \quad , \tag{2}$$

where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2$ is the Einstein tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ are the five-dimensional Ricci tensor and scalar respectively, and $\hat{g}_{AB}$ is the five-dimensional metric tensor. (Throughout this report capital Latin indices A, B, ... run over 0, 1, 2, 3, 4, and five-dimensional quantities are denoted by hats.) These equations can be derived by varying a five-dimensional version of the usual Einstein action:

$$S = \frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}d^4xdy \quad , \tag{3}$$

with respect to the five-dimensional metric, where $y = x^4$ represents the new (fifth) coordinate and $\hat{G}$ is a "five-dimensional gravitational constant."

The absence of matter sources in these equations reflects what we have emphasized as Kaluza's first key assumption (i), inspired by Einstein: that *the universe in higher dimensions is empty*. The idea is to explain matter (in four dimensions) as a manifestation of pure geometry (in higher ones). If, instead, one introduced new kinds of higher-dimensional matter, then one would have gained little in economy of thought. One would, so to speak, be getting Weinberg's "stone soup" [72] from a can.

3.2 A Minimal Extension of General Relativity

The five-dimensional Ricci tensor and Christoffel symbols are defined in terms of the metric exactly as in four dimensions:

$$\begin{aligned} \hat{R}_{AB} &= \partial_C\hat{\Gamma}^C_{AB} - \partial_B\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \quad , \\ \hat{\Gamma}^C_{AB} &= \frac{1}{2}\hat{g}^{CD}(\partial_A\hat{g}_{DB} + \partial_B\hat{g}_{DA} - \partial_D\hat{g}_{AB}) \quad . \end{aligned} \tag{4}$$

Note that, aside from the fact that tensor indices run over 0-4 instead of 0-3, all is exactly as it was in Einstein's theory. We have emphasized this as the second key feature (*ii*) of Kaluza's approach to unification.

Everything now depends on one's choice for the form of the five-dimensional metric. In general, one identifies the $\alpha\beta$-part of $\hat{g}_{AB}$ with $g_{\alpha\beta}$ (the four-dimensional metric tensor), the $\alpha 4$-part with A_α (the electromagnetic potential), and the 44-part with ϕ (a scalar field). A convenient way to parametrize things is as follows:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix} \quad , \tag{5}$$

where we have scaled the electromagnetic potential A_α by a constant κ in order to get the right multiplicative factors in the action later on. (Throughout this report, Greek indices α, β, ... run

over 0, 1, 2, 3, and small Latin indices a, b, ... run over 1, 2, 3. The four-dimensional metric signature is taken to be (+ - - -), and we work in units such that c = 1. In addition, for convenience and accord with other work, we set $\hbar = 1$ in §3, and $G = 1$ in in §7 and §8.)

3.3 The Cylinder Condition

If one then applies the third key feature (*iii*) of Kaluza's theory (the cylinder condition), which means dropping all derivatives with respect to the fifth coordinate, then one finds, using the metric (5) and the definitions (4), that the $\alpha\beta$-, $\alpha 4$-, and 44-components of the five-dimensional field equation (2) reduce respectively to the following field equations [15], [41] in four dimensions:

$$\begin{aligned} G_{\alpha\beta} &= \frac{\kappa^2\phi^2}{2}T^{EM}_{\alpha\beta} - \frac{1}{\phi}\Big[\nabla_\alpha(\partial_\beta\phi) - g_{\alpha\beta}\Box\phi\Big] \\ \nabla^\alpha F_{\alpha\beta} &= -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta} \quad , \quad \Box\phi = \frac{\kappa^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta} \quad , \end{aligned} \tag{6}$$

where $G_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta}/2$ is the Einstein tensor, $T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F^\gamma_\alpha F_{\beta\gamma}$ is the electromagnetic energy-momentum tensor, and $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. There are a total of 10 + 4 + 1 = 15 equations, as expected since there are fifteen independent elements in the five-dimensional metric (5).

3.4 The Case ϕ = constant

If the scalar field ϕ is constant throughout spacetime, then the first two of eqs. (6) are just the Einstein and Maxwell equations:

$$G_{\alpha\beta} = 8\pi G\phi^2 T^{EM}_{\alpha\beta} \quad , \quad \nabla^\alpha F_{\alpha\beta} = 0 \quad , \tag{7}$$

where we have identified the scaling parameter κ in terms of the gravitational constant G (in four dimensions) by:

$$\kappa \equiv 4\sqrt{\pi G} \quad . \tag{8}$$

This is the result originally obtained by Kaluza and Klein, who set $\phi = 1$. (The same thing has done by some subsequent authors employing "special coordinate systems" [9], [195].) The condition ϕ = constant is, however, only consistent with the the *third* of the field equations (6) when $F_{\alpha\beta}F^{\alpha\beta} = 0$, as was first pointed out by Jordan [39], [40] and Thiry [41]. The fact that this took twenty years to be acknowledged is a measure of the deep suspicion with which scalar fields were viewed in the first half of this century.

Nowadays the same derivation is usually written in variational language. Using the metric (5) and the definitions (4), and invoking the cylinder condition not only to drop derivatives with respect to y, but also to pull $\int dy$ out of the action integral, one finds that eq. (3) contains three components [10]:

$$S = -\int d^4x\sqrt{-g}\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{3\kappa^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right) \quad , \tag{9}$$

where G is defined in terms of its five-dimensional counterpart $\hat{G}$ by:

$$G \equiv \hat{G} / \int dy \quad , \tag{10}$$

and where we have used equation (8) to bring the factor of $16\pi G$ inside the integral. As before, if one takes $\phi =$ constant, then the first two components of this action are just the Einstein-Maxwell action for gravity and electromagnetic radiation (scaled by factors of ϕ). The third component is the action for a massless Klein-Gordon scalar field.

The fact that the action (3) leads to (9), or - equivalently - that the sourceless field equations (2) lead to (6) *with* source matter, constitutes the central miracle of Kaluza-Klein theory. Four-dimensional matter (electromagnetic radiation, at least) has been shown to arise purely from the geometry of empty five-dimensional spacetime. The goal of all subsequent Kaluza-Klein theories has been to extend this success to *other* kinds of matter.

3.5 The Case $A_\alpha = 0$: Brans-Dicke Theory

If one does *not* set $\phi =$ constant, then Kaluza's five-dimensional theory contains besides electromagnetic effects a Brans-Dicke-type scalar field theory, as becomes clear when one considers the case in which the electromagnetic potentials vanish, $A_\alpha = 0$. Without the cylinder condition, this would be no more than a choice of coordinates, and would not entail any loss of algebraic generality. (It would be exactly analogous to the common procedure in ordinary electrodynamics of choosing four-space coordinates in which either the electric or magnetic field disappears.) With the cylinder condition, however, we are effectively working in a special set of coordinates, so that the theory is no longer invariant with respect to general (ie., five-dimensional) coordinate transformations. The restriction $A_\alpha = 0$ is, therefore, a physical and not merely mathematical one, and restricts us to the "graviton-scalar sector" of the theory.

This is acceptable in some contexts - in homogeneous and isotropic situations, for example, where off-diagonal metric coefficients would pick out preferred directions; or in early-universe models which are dynamically dominated by scalar fields. Neglecting the A_α-fields, then, eq. (5) becomes:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \phi^2 \end{pmatrix} \quad , \tag{11}$$

With this metric, the field equations (2), and Kaluza's assumptions (i) - (iii) as before, the action (3) reduces to:

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R\phi \quad . \tag{12}$$

This is the special case $\omega = 0$ of the Brans-Dicke action [196]:

$$S_{BD} = -\int d^4x \sqrt{-g} \left(\frac{R\phi}{16\pi G} + \omega \frac{\partial^\alpha \phi \partial_\alpha \phi}{\phi^2} \right) + S_m \quad , \tag{13}$$

where ω is the dimensionless Brans-Dicke constant and the term S_m refers to the action associated with any matter fields which may be coupled to the metric or scalar field.

The value of ω is of course constrained to be greater than $\sim$500 by observation [197], so this simple model is certainly not viable, in the present era at least. One can however evade this limit by adding a nonzero potential V(ϕ) to the above action, as in extended inflation [198] and other theories [199], [200]; or by allowing the Brans-Dicke parameter ω to vary as a function of ϕ, as in hyperextended [201] and other inflationary models [202].

3.6 Conformal Rescaling

One can also re-formulate the problem by carrying out a Weyl, or *conformal rescaling* of the metric tensor. Conformal factors have begun to appear frequently in papers on Kaluza-Klein theory, but they have as yet received little attention in reviews of the subject, so we will discuss them briefly here, referring the reader to the literature for details.

The extra factor of ϕ in the action (9) above implies that, strictly speaking, the scalar field would have to be constant throughout spacetime [10], [13] in order for the gravitational part of the action to be in canonical form. Some authors [9], [195] have in fact set it equal to one by definition, though this is of course not a generally covariant procedure. The offending factor can however be removed by conformally rescaling the five-dimensional metric:

$$\hat{g}_{AB} \rightarrow \hat{g}'_{AB} = \Omega^2 \hat{g}_{AB} \quad , \tag{14}$$

where $\Omega^2 > 0$ is the conformal (or Weyl) factor, a function of the first four coordinates only (assuming Kaluza's cylinder condition). This is one step removed from the simplest possible realization of Kaluza's idea. In compactified and projective theories, however, there can be no physical objection to such a procedure since it takes place "in higher dimensions" which are not accessible to observation. Questions only arise in the process of dimensional reduction; ie., in interpreting the "real," four-dimensional quantities in terms of the rescaled five-dimensional ones.

The four-dimensional metric tensor is rescaled by the same factor as the five-dimensional one ($g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$), and this has the following effect on the four-dimensional Ricci scalar [203]:

$$R \rightarrow R' = \Omega^{-2}\left(R + 6\frac{\Box\Omega}{\Omega}\right) \quad . \tag{15}$$

A convenient parametrization is obtained by making the trivial redefinition $\phi^2 \rightarrow \phi$ and then introducing the conformal factor $\Omega^2 = \phi^{-1/3}$, so that the five-dimensional metric reads:

$$(\hat{g}'_{AB}) = \phi^{-1/3} \begin{pmatrix} g_{\alpha\beta} + \kappa^2 \phi A_\alpha A_\beta & \kappa\phi A_\alpha \\ \kappa\phi A_\beta & \phi \end{pmatrix} \quad , \tag{16}$$

The same procedure as before then leads [8], [12], [13] to the following *conformally rescaled* action instead of eq. (9) above:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{4}\phi F'_{\alpha\beta}F'^{\alpha\beta} + \frac{1}{6\kappa^2}\frac{\partial'^\alpha\phi\partial'_\alpha\phi}{\phi^2}\right) \quad , \tag{17}$$

where primed quantities refer to the rescaled metric (ie., $\partial'^\alpha\phi = g'^{\alpha\beta}\partial_\beta\phi$), and G and κ are defined as before. The gravitational part of the action then has the conventional form, as desired.

The Brans-Dicke case, obtained by putting $A_\alpha = 0$ in the metric, is also modified by the presence of the conformal factor. One finds (again making the redefinition $\phi^2 \rightarrow \phi$ and using $\Omega^2 = \phi^{-1/3}$) that the action (12) becomes [13]:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{6\kappa^2}\frac{\partial'^\alpha\phi\partial'_\alpha\phi}{\phi^2}\right) \quad , \tag{18}$$

In terms of the "dilaton" field $\sigma \equiv ln\phi/(\sqrt{3}\kappa)$, this action can be written:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{2}\partial'^\alpha\sigma\partial'_\alpha\sigma\right) \quad , \tag{19}$$

which is the canonical action for a minimally coupled scalar field with no potential [204].

3.7 Conformal Ambiguity

The question of conformal ambiguity arises when we ask, "Which is the real four-dimensional metric (ie., the one responsible for Einstein's gravity) - the original $g_{\alpha\beta}$, or the rescaled $g'_{\alpha\beta}$?" The issue was already raised at least as far back as 1955 by Pauli [119]. (The rescaled metric is sometimes referred to in the literature as the "Pauli metric," as opposed to the unrescaled "Jordan metric.") Most authors have worked with the traditional (unrescaled) metric, if indeed they have troubled themselves over the matter at all [205]. Others [206], [207] have considered the interesting idea of coupling visible matter (including that involved in the classical tests of general relativity) to the Jordan metric, but allowing *dark matter* to couple to a rescaled Pauli metric. In recent years, a variety of new arguments have been advanced in favor of regarding the rescaled metric as the true "Einstein metric" for *all* types of matter in compactified Kaluza-Klein theory. The following paragraph is intended as a brief review of these; many are discussed more thoroughly in [205].

The first use of conformal rescaling to pick out "physical fields" was in certain ten-dimensional supergravity [208] and superstring [209] models of the early 1980s. It then appeared in work on the quantum aspects of Kaluza-Klein theory [210], and on the stability of compactified Kaluza-Klein cosmologies [211], [212]. In these papers it was asserted that conformal ambiguity affected the physics at the quantum but not the classical level. This was supported by a demonstration [213] that the mass of a five-dimensional Kaluza-Klein monopole was invariant with respect to conformal rescaling, although it was speculated in this paper that the addition of matter fields would complicate the situation. Cho [207] confirmed this suspicion by showing explicitly that the conformal invariance of the Brans-Dicke action (13) would be broken for $S_m = 0$. This resulted in different matter couplings to the metric for different conformal factors, which would manifest themselves as "fifth force"-type violations of the weak equivalence principle [214]. He argued in addition that only one conformal factor - the factor $\phi^{-1/3}$ used above - could allow one to properly interpret the metric as a massless spin-two graviton [207]; and moreover that without this factor the kinetic energy of the scalar field would be unbounded from below, making the theory unstable [61]. This last point has also been emphasized by Sokolowski and others [205], [215], [216]. (Note that the conformal factor $\sqrt{\varphi}$ used by these authors is the same as the one discussed above; the exponent depends on whether one rescales the five-, or only the four-dimensional metric. The scalar φ is related to ϕ simply by $\varphi = \phi^{-3/2}$.) It has also been claimed that conformal rescaling is necessary in scale-invariant Kaluza-Klein cosmology [217] if one is to properly interpret the effective four-dimensional Friedmann-Robertson-Walker scale factor. A recent discussion of conformal ambiguity in compactified Kaluza-Klein theory is found in [218].

There is also something much like a conformal rescaling of coordinates in projective Kaluza-Klein theory, notably in the work of Schmutzer after 1980 [16], [17], [127], [128], where it is introduced in order to eliminate unwanted second-order scalar field terms from the generalized gravitational field equations. In *noncompactified* Kaluza-Klein theory, by contrast, there has been no discussion of conformal rescaling. This is largely because the extra dimensions are regarded as *physical* (if not necessarily lengthlike or timelike). The five-dimensional metric, in effect, becomes accessible (in principle) to observation, and conformally transforming it at will may no longer be so innocuous. We will not consider the issue further in this report; interested readers are directed to Sokolowski's paper [205].

4 Compactified Theories

So far we have introduced Kaluza's theory, with its cylinder condition, but have deliberately postponed discussion of compactification because we wish to emphasize that it is logically distinct from cylindricity, and in particular that it is only one mechanism by which to explain the apparently four-dimensional nature of the world. We now turn to compactified Kaluza-Klein theory, but keep our discussion short as this subject has been thoroughly reviewed elsewhere ([3]-[7], [8]-[14], [195]).

4.1 Klein's Compactification Mechanism

The somewhat contrived nature of Kaluza's assumption, that a fifth dimension exists but that no physical quantities depend upon it, has struck generations of unified field theorists as inadequate. Klein arrived on the scene during the tremendous excitement surrounding the birth of quantum theory, and perhaps not surprisingly had the idea [2], [36] of explaining the lack of dependence by making the extra dimension very small. (The story that this was suggested to him on hearing a colleague address him by his last name has, so far as we know, no basis in historical fact.)

Klein assumed that the fifth coordinate was to be a lengthlike one (like the first three), and assigned it two properties: (1) a *circular topology* (S^1); and (2) a *small scale*. Under property (1), any quantity $f(x,y)$ (where x = (x^0, x^1, x^2, x^3) and $y = x^4$) becomes *periodic*; $f(x,y) = f(x, y+2\pi r)$ where r is the scale parameter or "radius" of the fifth dimension. Therefore all the fields can be Fourier-expanded:

$$g_{\alpha\beta}(x,y) = \sum_{n=-\infty}^{n=\infty} g^{(n)}_{\alpha\beta}(x) e^{iny/r} \quad , \quad A_\alpha(x,y) = \sum_{n=-\infty}^{n=\infty} A^{(n)}_{\alpha}(x) e^{iny/r} \quad , \tag{20}$$

$$\phi(x,y) = \sum_{n=-\infty}^{n=\infty} \phi^{(n)} e^{iny/r} \quad ,$$

where the superscript $^{(n)}$ refers to the nth Fourier mode. Thanks to quantum theory, these modes carry a momentum in the y-direction of the order $|n|/r$. This is where property (2) comes in: if r is small enough, then the y-momenta of even the $n = 1$ modes will be so large as to put them beyond the reach of experiment. Hence only the $n = 0$ modes, which are independent of y, will be observable, as required in Kaluza's theory.

How big could the scale size r of a fourth spatial dimension be? The strongest constraints have come from high-energy particle physics, which probes increasingly higher mass scales and correspondingly smaller length scales (the Compton wavelength of massive modes is of the order M^{-1}). Experiments of this kind [55] presently constrain r to be less than an attometer in size ($1am = 10^{-18}m$). Theorists often set r equal to the Planck length $\ell_{pl} \sim 10^{-35}m$, which is both a natural value and small enough to guarantee that the mass of any $n = 0$ Fourier modes lies beyond the Planck mass $m_{pl} \sim 10^{19}$ GeV.

In general, one identifies Kaluza's five-dimensional metric (5) with the full (Fourier-expanded) metric $\hat{g}_{AB}(x,y)$, higher modes and all. One then makes what is known in compactified theory as the "Kaluza-Klein ansatz," which consists in discarding all massive ($n \neq 0$) Fourier modes, as justified above. In the five-dimensional case, the Kaluza-Klein ansatz amounts to simply dropping the y-dependency of $g_{\alpha\beta}$, A_α, and ϕ, giving the effective four-dimensional "low-energy" theory of the graviton $g^{(0)}_{\alpha\beta}$, photon $A^{(0)}_\alpha$ and scalar $\phi^{(0)}$. For higher dimensions, though, the relationship between the full metric and the metric obtained with the "Kaluza-Klein ansatz" is more complicated, as has

been emphasized by Duff et al. [11], [12]. These authors also stress the difference between these two metrics and a third important metric in Kaluza-Klein theory, the ground state metric $\langle \hat{g}_{AB} \rangle$ which is the vacuum expectation value of the full metric $\hat{g}_{AB}(x, y)$, and determines the topology of the compact space. In the five-dimensional case described above, which is topologically $M^4 \times S^1$, this looks like:

$$(\langle \hat{g}_{AB} \rangle) = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{21}$$

where $\eta_{\alpha\beta}$ is the four-dimensional Minkowski space metric.

4.2 Quantization of Charge

The expansion of fields into Fourier modes suggests a possible mechanism to explain *charge quantization*, and it is interesting to see what became of this idea [8]. One begins by introducing five-dimensional matter into the theory, leaving aside for the moment questions as to what this would correspond to physically. The simplest kind of matter is a massless five-dimensional scalar field $\hat{\psi}(x, y)$. Its action would have a kinetic part only:

$$S_{\hat{\psi}} = -\int d^4x dy \sqrt{-g} \partial^A \hat{\psi} \partial_A \hat{\psi} \quad . \tag{22}$$

The field can be expanded like those in eq. (20):

$$\hat{\psi}(x, y) = \sum_{n=-\infty}^{n=\infty} \hat{\psi}^{(n)} e^{iny/r} \quad . \tag{23}$$

When this expansion is put into the action (22), one finds (using eq. (16)) the following result [8], analogous to eq. (9):

$$S_{\hat{\psi}} = -\left(\int dy \right) \sum_n \int d^4x \sqrt{-g} \quad \Bigg[\quad \left(\partial^\alpha + \frac{in\kappa A^\alpha}{r} \right) \hat{\psi}^{(n)} \left(\partial_\alpha + \frac{in\kappa A_\alpha}{r} \right) \hat{\psi}^{(n)} - \frac{n^2}{\phi r^2} \hat{\psi}^{(n)2} \Bigg] \quad , \tag{24}$$

From this action one can read off both the charge and mass of the scalar modes $\hat{\psi}^{(n)}$. Comparison with the minimal coupling rule $\partial_\alpha \to \partial_\alpha + ieA_\alpha$ of quantum electrodynamics (where e is the electron charge) shows that in this theory the *nth* Fourier mode of the scalar field $\hat{\psi}$ also carries a *quantized charge*:

$$q_n = \frac{n\kappa}{r} \left(\phi \int dy \right)^{-1/2} = \frac{n\sqrt{16\pi G}}{r\sqrt{\phi}} \quad , \tag{25}$$

where we have normalized the definition of A_α in the action (17) by dividing out the factor $(\phi \int dy)^{1/2}$, and made use of the definitions (8) and (10) for κ and G respectively. As a corollary to this result one can also come close to *predicting* the value of the fine structure constant, simply by identifying the charge q_1 of the first Fourier mode with the electron charge e. Taking $r\sqrt{\phi}$ to be on the order of the Planck length $\ell_{pl} = \sqrt{G}$, one has:

$$\alpha \equiv \frac{q_1^2}{4\pi} \sim \frac{(\sqrt{16\pi G}/\sqrt{G})^2}{4\pi} = 4 \quad . \tag{26}$$

(An improved determination of $r\sqrt{\phi}$ would presumably hit closer to the mark.) The possibility of thus explaining an otherwise "fundamental constant" would have made compactified five-dimensional Kaluza-Klein theory very attractive.

However, the *masses* of the scalar modes are not at all compatible with these ideas. These are given by the square root of the coefficient of the $\hat{\psi}^{(n)2}$-term:

$$m_n = \frac{|n|}{r\sqrt{\phi}} \quad . \tag{27}$$

If $r\sqrt{\phi} \sim \ell_{pl}$ as we assumed, then the electron mass m_1 (corresponding to the first Fourier mode) would be ℓ_{pl}^{-1}; ie., the Planck mass $m_{pl} \sim 10^{19}$ GeV, rather than 0.5 MeV. This discrepancy of some twenty-two orders of magnitude between theory and observation played a large role in the abandonment of five-dimensional Kaluza-Klein theory.

In modern compactified theories, one avoids this problem by doing three things [8]: (1) identifying observed (light) particles like the electron with the $n = 0$, rather than the higher modes of the Fourier expansion above. From eq. (27), these particles therefore have zero mass at the level of the field equations. However, one then invokes: (2) the mechanism of spontaneous symmetry-breaking to bestow on them them the modest masses required by observation. From eq. (25) above, there is also the problem of explaining how the $n = 0$ modes can have nonzero *charge* (or, more generally, nonzero couplings to the gauge fields). This is solved by: (3) going to higher dimensions, where massless particles are no longer "singlets of the gauge group" corresponding to the ground state (eg., the 44-part of the metric (21) above). We look at this procedure briefly in the next section.

The other way to avoid the problems of compactified five-dimensional Kaluza-Klein theory is, of course, to look at projective theories, or indeed to loosen the restriction of compactification on the fifth dimension altogether. These approaches probably mean giving up the ready-made explanation for charge quantization described above.

4.3 Extension to Higher Dimensions

The key to extending the Kaluza-Klein formalism to strong and weak nuclear interactions lies in recognizing that electromagnetism has been effectively incorporated into general relativity by adding $U(1)$ local gauge invariance to the theory, in the form of local coordinate invariance with respect to $y = x^4$. Assuming the extra coordinate has a circular topology and a small scale, the theory is invariant under transformations:

$$y \to y' = y + f(x) \quad , \tag{28}$$

where x stands for the four-space coordinates x^0, x^1, x^2, x^3. With the aid of the usual tensor transformation law (in *five* dimensions):

$$\hat{g}_{AB} \to \hat{g}'_{AB} = \frac{\partial x^C}{\partial x'^A}\frac{\partial x^D}{\partial x'^B}\hat{g}_{CD} \quad , \tag{29}$$

one then finds that the only change to the metric (5) is:

$$A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha f(x) \quad , \tag{30}$$

which is just a $U(1)$ local gauge transformation. In other words the theory is locally $U(1)$ gauge invariant. It is thus not surprising that electromagnetism could be contained in five-dimensional general relativity.

To extend the same approach to more complicated symmetry groups, one goes to higher dimensions [8]. The metric corresponding to the "Kaluza-Klein ansatz" ($n = 0$ modes only) can be written (cf. eq. (5)):

$$\left(\hat{g}^{(0)}_{AB}\right) = \begin{pmatrix} g_{\alpha\beta} + \tilde{g}_{\mu\nu} K^\mu_i A^i_\alpha K^\nu_j A^j_\beta & \tilde{g}_{\mu\nu} K^\mu_i A^i_\alpha \\ \tilde{g}_{\mu\nu} K^\nu_i A^i_\beta & \tilde{g}_{\mu\nu} \end{pmatrix} \quad , \tag{31}$$

where $\tilde{g}_{\mu\nu}$ is the metric of the d-dimensional *compact* space. Indices μ, ν, ... run from 1 to d, while A, B, ... run from 0 to $(3 + d)$, and α, β, ... run from 0 to 3 as usual. The K^μ_i are a set of n linearly independent Killing vectors for the compact manifold ($i = 1, \ldots, n$). Analogously to eq. (28) one then assumes that the theory is locally invariant under transformations:

$$y^\mu \to y'^\mu = y^\mu + \sum_{i=1}^{n} \varepsilon^i(x) K^\mu_i \quad , \tag{32}$$

where the $\varepsilon^i(x)$ are a set of n infinitesimal parameters. Because Killing vectors by definition satisfy:

$$\frac{\partial K^\lambda_i}{\partial y^\mu} \tilde{g}_{\lambda\nu} + \frac{\partial K^\lambda_i}{\partial y^\nu} \tilde{g}_{\lambda\mu} + K^\lambda_i \frac{\partial \tilde{g}_{\mu\nu}}{\partial y^\lambda} = 0 \quad , \tag{33}$$

the transformation law (29) leaves the $\tilde{g}_{\mu\nu}$-part of the metric untouched, and the only effect on eq. (31) is:

$$A^i_\alpha \to A'^i_\alpha = A^i_\alpha + \partial_\alpha \varepsilon^i(x) \quad , \tag{34}$$

which is a local gauge transformation whose gauge group is the *isometry group* (G, say) of the compact manifold. Thus one might hope that higher-dimensional general relativity could contain *any* gauge theory.

The larger symmetry of the higher-dimensional mechanism also allows for nonzero couplings of the $n = 0$ modes to the gauge fields; ie., for "charged" massless particles (which, as we saw, was impossible in the five-dimensional case). Massless scalar fields $\phi_a(x)$ in the adjoint representation of the gauge group, for example, can be introduced [8] via:

$$\Phi^\mu_a = \phi_a(x) K^\mu_a(y) \quad , \tag{35}$$

and these in general have nonzero couplings to the gauge fields because the $K^\mu_a(y)$ are not covariantly constant.

4.4 Higher-Dimensional Matter

It is crucial to realize, however, that the above "ansatz" metric (31) does *not* satisfy Einstein's equations in $4 + d$ dimensions unless the Killing vectors are independent of $\{y\}$, the extra coordinates [8] - ie., unless the compact manifold is *flat* [7]. The ground state metric (cf. eq. (21)) is:

$$(\langle \hat{g}_{AB} \rangle) = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \tilde{g}_{\mu\nu}(y) \end{pmatrix} \quad . \tag{36}$$

The vacuum Einstein equations are $\hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2 = 0$. Since $\langle \hat{g}_{\alpha\beta} \rangle = \eta_{\alpha\beta}$ is flat, $\hat{R}_{\alpha\beta} = 0$. Therefore, from the $\alpha\beta$-components of the field equations, $\hat{R}$ must vanish. But then the $\mu\nu$-parts of the same equations imply that $\hat{R}_{\mu\nu} = 0$; ie., that $\tilde{g}_{\mu\nu}$ must also be flat. In what is perhaps a symptom of the split that has developed since Klein between the particle physics and general relativity sides of higher-dimensional unification research, early workers tended to ignore this "consistency problem" [11], [12], and placed no restrictions on the compact manifold while continuing to use the metric (31). Recently Cho [61]-[64] has raised related questions about whether the "zero modes" might not become massive (and $\{y\}$-dependent) in the event of spontaneous symmetry-breaking, and has even suggested "kicking away the ladder" of Klein's Fourier modes entirely, basing dimensional reduction a *priori* on isometry instead.

It is now widely recognized [13] that conventional compactification of d extra spatial dimensions (where $d > 1$) requires either (1) explicit higher-dimensional matter terms, which can induce "spontaneous compactification" by imposing constant curvature on the compact manifold [70], [71]; or (2) other modifications of the higher-dimensional theory, such as the inclusion of torsion [65]-[68] or higher-derivative (eg. R^2) terms [69]. Most higher-dimensional compactified Kaluza-Klein theories rely on higher-dimensional matter of one kind or another. For example, in Freund-Rubin compactification [86], which is the basis of eleven-dimensional supergravity, one introduces a third-rank antisymmetric tensor field $\hat{A}_{BCD}$ with field strength:

$$\hat{F}_{ABCD} \equiv \partial_A \hat{A}_{BCD} - \partial_B \hat{A}_{ACD} + \partial_C \hat{A}_{ABD} - \partial_D \hat{A}_{ABC} \quad , \tag{37}$$

and free action given by:

$$S_{\hat{A}} = -\frac{1}{384\pi G} \int d^{(4+d)}x \sqrt{-g} \hat{F}_{ABCD} \hat{F}^{ABCD} \quad . \tag{38}$$

The effect of this [13] is to add an explicit energy-momentum tensor to the right-hand side of the higher-dimensional Einstein equations (1):

$$\hat{T}_{AB} = -\frac{1}{48\pi G} \left(\hat{F}_{CDEA} \hat{F}_B^{CDE} - \frac{1}{8} \hat{F}_{CDEF} \hat{F}^{CDEF} \hat{g}_{AB} \right) \quad , \tag{39}$$

The matter fields required to achieve compactification are not the end of the story, however. Others are in general needed if the theory is to contain the full gauge group of the standard model (including strong and weak interactions). Witten [84] has shown that this requires a theory of at least eleven dimensions (including the four macroscopic ones). While there are an infinite number of compact seven-dimensional manifolds whose isometry groups $G \supset SU(3) \times SU(2) \times U(1)$, none of them give rise to realistic quark and lepton representations [8], [13]. It is possible to obtain quarks and leptons from other manifolds such as the 7-sphere and the "squashed" 7-sphere [12]. The symmetry groups of these manifolds ($SO(8)$ and $SO(5) \times SU(2)$ respectively) are, however, not large enough to contain the standard model, and additional "composite" matter fields [13], [91] are therefore required.

Explicit higher-dimensional fields may also be required to incorporate *chirality* into eleven-dimensional compactified theory [7], [73], [77] (this is difficult in an odd number of dimensions). Two other schemes by which this might be accomplished are modifications of Riemannian geometry [100]-[102] and *noncompact* internal manifolds [93]-[99]. Thus in the $D = 11$ "chirality problem" one finds again a choice between sacrificing either (*i*) the equation "Matter=Geometry;" (*ii*) the geometrical basis of Einstein's theory; or (*iii*) cylindricity. Compactified theory has in general been characterized by a readiness to drop (*i*).

We conclude this section by noting that the situation (with regard to non-geometrized matter) does not improve in ten-dimensional compactified theory; in fact, in many cases a six-dimensional

internal manifold with *no* isometries is used [8], which means that *all* the matter is effectively put in by hand, marking a complete abandonment of the original Kaluza programme. Besides ensuring compactification and making room for fermions, extra terms in the ten-dimensional Lagrangian also play a role in suppressing *anomalies*. In the two most popular $D = 10$ theories, for example (those based on the symmetry groups $SO(32)$ [104] and $E_8 \times E_8$ [105]), this is accomplished by Chapline-Manton terms [8]. To the extent that these terms arise naturally in the low-energy limit of ten-dimensional *superstring* theory, however, they are less arbitrary than some of the others we have mentioned. There is no doubt that superstrings currently offer, within the context of compactified Kaluza-Klein theory, the best hope for a unified "theory of everything" [89], [108]. Whether the compactified approach is the best one remains - as we hope to show in the rest of this report - an open question.

5 Projective Theories

Projective theories were designed to emulate the successes of Kaluza's five-dimensional theory without the epistemological burden of a real fifth dimension. Early models did this too well: like Kaluza's (with no dependence on the fifth coordinate and no added "higher-dimensional matter" fields) they gave back $\omega = 0$ Brans-Dicke theory when the electromagnetic potentials were switched off. This contradicted time-delay measurements like that of the Martian Viking lander [197]. There were other problems as well [16]. Modern projective theories [15]-[18] attempt to overcome these shortcomings in at least two different ways.

5.1 A Theory of Elementary Particles

Lessner [15] has suggested that, although experiments rule out a macroscopic Brans-Dicke-type scalar field, the theory might still be applicable on *microscopic* scales, and could be used to describe the internal structure of elementary particles. He begins with the same five-dimensional field equations (2) (now interpreted as *projector equations*), and obtains the same four-dimensional field equations (6), except that the constant G is replaced by "B," which becomes essentially a free parameter of the theory. A solution ($g_{\alpha\beta}$, $F_{\alpha\beta}$, ϕ) of the field equations is called a "particle" if it satisfies certain conditions on symmetry, positivity and asymptotic behaviour [15]. Some of the properties of these particles are explored in [123]-[126]. The theory is only applicable to *macroscopic* phenomena when $\phi = 1$ and the third of the field equations (6) is omitted.

5.2 Projective Unified Field Theory

Schmutzer has taken an alternate approach since 1980 in his "projective unified field theory" or PUFT [16]-[18] by explicitly introducing "non-geometrizable matter" (the so-called "substrate"). In ordinary Kaluza-Klein theory this would correspond to higher-dimensional matter and be represented in the five-dimensional Einstein equations (1) by a nonzero energy-momentum tensor $\hat{T}_{AB}$; in the projective theory one has instead an *energy projector* $\hat{\theta}_{AB}$:

$$\hat{G}_{AB} = 8\pi\hat{G}\hat{\theta}_{AB} \quad . \tag{40}$$

There is also a conformal rescaling of the four-dimensional metric, as mentioned earlier:

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = e^{-\sigma} g_{\alpha\beta} \quad , \tag{41}$$

where σ is a new scalar field. Eqs. (40) break down, analogously to the five-dimensional ones (2), to the following set of equations in four dimensions:

$$\begin{aligned} G_{\alpha\beta} &= 8\pi G(T^{EM}_{\alpha\beta} + \Sigma_{\alpha\beta} + \theta_{\alpha\beta}) \qquad , \quad \nabla_\alpha H^{\alpha\beta} = J^\beta \quad , \\ \Box\sigma &= 8\pi G\left(\frac{2}{3}\vartheta + \frac{1}{2}B_{\alpha\beta}H^{\alpha\beta}\right) \qquad\qquad\qquad , \end{aligned} \tag{42}$$

where $T^{EM}_{\alpha\beta}$ is the electromagnetic energy-momentum tensor as before, and where there are also two new energy-momentum tensors: the *substrate energy tensor* $\theta_{\alpha\beta} = \hat{\theta}_{\alpha\beta}$ and the *scalaric energy tensor* $\Sigma_{\alpha\beta}$ defined by:

$$\Sigma_{\alpha\beta} = -\frac{3}{16\pi G}\left(\partial_\alpha\sigma\partial_\beta\sigma - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\sigma\partial_\gamma\sigma\right) \quad . \tag{43}$$

The other terms in eqs. (42) are the electric four-current density J_α, the electromagnetic field strength tensor $B_{\alpha\beta}$, the induction tensor $H_{\alpha\beta} = e^{3\sigma}B_{\alpha\beta}$ (the factor $e^{3\sigma}$ acts here as a kind of "scalaric dielectricity"), and one more new quantity, the *scalaric substrate density*:

$$\vartheta = e^{-\sigma}\hat{\theta}^A_A - \frac{3}{2}\theta^\alpha_\alpha \quad . \tag{44}$$

The conservation of energy $\hat{\nabla}^A\hat{\theta}_{AB} = 0$ implies not only conservation of four-current ($\nabla_\alpha J^\alpha = 0$) but also conservation of *substrate energy*:

$$\nabla_\beta\theta^{\alpha\beta} = -B^\alpha_\beta J^\beta + \vartheta\nabla^\alpha\sigma \quad . \tag{45}$$

The existence of substrate and scalaric matter in PUFT gives rise to phenomena such as "scalaric polarization" of the vacuum, and violations of the weak equivalence principle for time-dependent scalaric fields. These can be quantified in terms of a "scalarism parameter" γ, defined as the ratio of scalaric substrate density to the density ρ of ordinary matter:

$$\gamma \equiv \vartheta/\rho \quad . \tag{46}$$

This number becomes in practice the primary free parameter of the theory, showing up in PUFT-based cosmological models [127], [129], equivalence principle-type experiments [128], and Solar System tests (perihelion shift, light deflection and time delay) [17]. Experimental constraints on the theory take the form of upper limits on the size of γ.

In comparing these projective theories to the compactified Kaluza-Klein theories of the last section, one could perhaps summarize as follows: Kaluza's unified theory as it stands is an elegant (no higher-dimensional matter) and minimal extension of general relativity, but suffers from the defect of a very contrived-looking cylinder condition. Five-dimensional compactified theory, beginning with Klein, repairs this flaw (and even offers the possibility of explaining charge quantization) but turns out to disagree radically with observation. To overcome this problem within the context of compactified theory, one has to go to higher dimensions and either introduce higher-dimensional matter or higher-derivative terms to the Einstein action, if one wishes to obtain satisfactory compactification. Projective theory offers an alternative way to "explain" the cylinder condition, and can (unlike compactified theory) be formulated in a way that is compatible with experiment using only one extra "dimension." This comes at the price, however, not only of modifying the geometrical foundation of Einstein's theory, but (in Schmutzer's case) of introducing a "non-geometrizable substrate," or

(in Lessner's case) of limiting one's ambitions to microscopic phenomena. Overall, the projective approach does not appear to us to be an improvement over compactified theory.

6 Noncompactified Theories

An alternative is to stay with the idea that the new coordinates are physical, but to *generalize* the compactified approach by relaxing the cylinder condition [19]-[26], instead of restricting the topology and scale of the fifth dimension in an attempt to satisfy it exactly. This means that physical quantities, including in particular those derived from the metric tensor, will depend on the fifth coordinate. In fact it is precisely this dependence which allows one to obtain not only electromagnetic radiation, but matter of a very *general* kind from geometry via the higher-dimensional field equations. The equations of motion, too, are modified by dependence on extra coordinates. We review these facts in the next few sections.

Of course, the fifth dimension might also be expected to appear elsewhere in physics, and one of the primary challenges of noncompactified theory is to explain why its effects have not been noticed so far. Why, for example, have experiments such as those mentioned earlier [55] been able to restrict the size of any extra dimensions to below the attometer scale? In noncompactified theory, the answer is that extra coordinates are not necessarily *lengthlike*, as these experiments assume. Following Minkowski's example, one can imagine coordinates of other kinds, scaled by appropriate dimension-transposing parameters (like c) to give them units of length. We review this important issue, and the evidence for the hypothesis that a fifth dimension might be physically related to *rest mass*, at the end of §6. For the moment, however, we put off questions of interpretation and begin by simply seeing how far Kaluza's five-dimensional unified field theory can be taken when the cylinder condition is dropped.

6.1 The Metric

Without cylindricity, there is no reason to compactify the fifth dimension, so this approach is properly called "noncompactified." Noncompact extra dimensions have also been considered in *compactified* Kaluza-Klein theory by Wetterich and others [93]-[99] as a way to bring chiral fermions into the theory and arrange for a vanishing four-dimensional cosmological constant. These authors, however, retain Klein's mechanism of harmonic expansion, which in turn means that the compact manifold must have finite volume. In the fully noncompactified approach we wish to make no *a priori* assumptions about the nature of the extra-dimensional manifold.

We begin with the same five-dimensional metric (5) as before, but choose coordinates such that the four components of A_α vanish. Since we are no longer imposing cylindricity on our solutions, this entails no loss of algebraic generality; it is analogous to the common strategy in electromagnetic theory of choosing coordinates such that either the electric or magnetic field vanishes. We also eschew any conformal factor here, preferring to treat the fifth dimension on the same footing as the other four. The five-dimensional metric tensor, then, is:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \varepsilon\phi^2 \end{pmatrix} \quad , \tag{47}$$

where we have introduced the factor ε in order to allow a timelike, as well as spacelike signature for the fifth dimension (we require only that $\varepsilon^2 = 1$).

Timelike extra dimensions are rarely considered in compactified Kaluza-Klein theory, for several reasons [13]: (1) they lead to the wrong sign for the Maxwell action in eq. (9) relative to the

Einstein one; and (2) they lead to the wrong sign for the mass m_n of the charged modes in eq. (24); ie., to the prediction of tachyons. The relevance of these two arguments to noncompactified theory may be debated. A third common objection (3) is that additional temporal [13] or timelike [7], [86] dimensions would lead to closed timelike curves and hence allow causality violation. One should be careful here to discriminate between temporal dimensions, which actually have physical units of time; and timelike ones, which merely have timelike signature. If the physical nature of the fifth coordinate were actually temporal, one could certainly imagine problems with causality. One can, however, transpose units with the proper combination of fundamental constants; changing a temporal one, for instance, into a spatial one with c. With regard to timelike extra dimensions, the situation is also less clear than is sometimes claimed. It has even been argued [219] that physics might be quite compatible with closed timelike curves. All in all, it is probably prudent to keep an open mind toward the signature of a physical fifth dimension.

6.2 The Field Equations

One now follows the same approach as Kaluza, using the same definitions (4) of the five-dimensional Christoffel symbols and Ricci tensor. Now, however, one *keeps* derivatives with respect to the fifth coordinate x^4 rather than assuming that they vanish. The resultant expressions for the $\alpha\beta$-, $\alpha 4$- and 44-parts of the five-dimensional Ricci tensor $\hat{R}_{AB}$ are [20]:

$$\begin{aligned}
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{\nabla_\beta(\partial_\alpha\phi)}{\phi} + \frac{\varepsilon}{2\phi^2}\left(\frac{\partial_4\phi\partial_4 g_{\alpha\beta}}{\phi} - \partial_4 g_{\alpha\beta}\right. \\
&\quad \left. + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right) \quad , \\
\hat{R}_{\alpha 4} &= \frac{g^{44}g^{\beta\gamma}}{4}(\partial_4 g_{\beta\gamma}\partial_\alpha g_{44} - \partial_\gamma g_{44}\partial_4 g_{\alpha\beta}) + \frac{\partial_\beta g^{\beta\gamma}\partial_4 g_{\gamma\alpha}}{2} \\
&\quad + \frac{g^{\beta\gamma}\partial_4(\partial_\beta g_{\gamma\alpha})}{2} - \frac{\partial_\alpha g^{\beta\gamma}\partial_4 g_{\beta\gamma}}{2} - \frac{g^{\beta\gamma}\partial_4(\partial_\alpha g_{\beta\gamma})}{2} \\
&\quad + \frac{g^{\beta\gamma}g^{\delta\epsilon}\partial_4 g_{\gamma\alpha}\partial_\beta g_{\delta\epsilon}}{4} + \frac{\partial_4 g^{\beta\gamma}\partial_\alpha g_{\beta\gamma}}{4} \quad , \\
\hat{R}_{44} &= -\varepsilon\phi\Box\phi - \frac{\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2} - \frac{g^{\alpha\beta}\partial_4(\partial_4 g_{\alpha\beta})}{2} \\
&\quad + \frac{\partial_4\phi g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2\phi} - \frac{g^{\alpha\beta}g^{\gamma\delta}\partial_4 g_{\gamma\beta}\partial_4 g_{\alpha\delta}}{4} \quad ,
\end{aligned} \tag{48}$$

where "$\Box$" is defined as usual (in four dimensions) by $\Box\phi \equiv g^{\alpha\beta}\nabla_\beta(\partial_\alpha\phi)$.

We assume that there is no "higher-dimensional matter," so the Einstein equations take the form (2), $\hat{R}_{AB} = 0$. The first of eqs. (48) then produces the following expression for the *four-dimensional* Ricci tensor:

$$\begin{aligned}
R_{\alpha\beta} &= \frac{\nabla_\beta(\partial_\alpha\phi)}{\phi} - \frac{\varepsilon}{2\phi^2}\left(\frac{\partial_4\phi\partial_4 g_{\alpha\beta}}{\phi} - \partial_4(\partial_4 g_{\alpha\beta})\right. \\
&\quad \left. + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right) \quad .
\end{aligned} \tag{49}$$

The second can be written in the form of a *conservation law*:

$$\nabla_\beta P^\beta_\alpha = 0 \quad , \tag{50}$$

where we have defined a new four-tensor by:

$$P^\beta_\alpha \equiv \frac{1}{2\sqrt{\hat{g}_{44}}}(g^{\beta\gamma}\partial_4 g_{\gamma\alpha} - \delta^\beta_\alpha \partial_4 g_{\gamma\epsilon}) \quad . \tag{51}$$

And the third of eqs. (48) takes the form of a *scalar wave equation* for ϕ:

$$\varepsilon\phi\Box\phi = -\frac{\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{4} - \frac{g^{\alpha\beta}\partial_4(\partial_4 g_{\alpha\beta})}{2} + \frac{\partial_4\phi g^{\alpha\beta}\partial_4 g_{\alpha\beta}}{2\phi} \quad . \tag{52}$$

Eqs. (49) - (52) form the basis of five-dimensional noncompactified Kaluza-Klein theory. It only remains to interpret their meaning in four dimensions, and then to apply them to any given physical problem by choosing the appropriate metric $\hat{g}_{AB}$. The rest of §6 is taken up with interpretation; applications to cosmology and astrophysics are the subjects of §7 and §8. We concentrate in this report on the five-dimensional case. The extension to arbitrary dimensions has yet to be investigated in detail, although some aspects of this have recently been discussed by Rippl, Romero & Tavakol [24]. (These authors also consider noncompactified *lower*-dimensional gravity, which might be more easily quantized than Einstein's theory).

6.3 Matter from Geometry

The best-understood of eqs. (49) - (52) is the first. It allows us to interpret *four-dimensional matter as a manifestation of five-dimensional geometry* [20]. One simply requires that the usual Einstein equations (with matter) hold in four dimensions:

$$8\pi G T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} \quad , \tag{53}$$

where $T_{\alpha\beta}$ is the matter energy-momentum tensor. Contracting eq. (49) with the metric $g^{\alpha\beta}$ gives (with the help of eq. (52)) the following expression for the four-dimensional Ricci scalar:

$$R = \frac{\varepsilon}{4\phi^2}\Big[\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta} + \Big(g^{\alpha\beta}\partial_4 g_{\alpha\beta}\Big)^2\Big] \quad . \tag{54}$$

Inserting this result, along with eq. (49), into eq. (53), one finds that:

$$\begin{aligned} 8\pi G T_{\alpha\beta} \;=\; & \frac{\nabla_\beta(\partial_\alpha\phi)}{\phi} - \frac{\varepsilon}{2\phi^2}\Bigg[\frac{\partial_4\phi\partial_4 g_{\alpha\beta}}{\phi} - \partial_4(\partial_4 g_{\alpha\beta}) + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} \\ & - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\Big(\partial_4 g^{\gamma\delta}\partial_4 g_{\gamma\delta} + (g^{\gamma\delta}\partial_4 g_{\gamma\delta})^2\Big)\Bigg] \quad . \end{aligned} \tag{55}$$

Provided we use this expression for $T_{\alpha\beta}$, the four-dimensional Einstein equations $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ are *automatically contained* in the five-dimensional vacuum ones $\hat{G}_{AB} = 0$. The matter described by $T_{\alpha\beta}$ is a manifestation of pure geometry in the higher-dimensional world. This has been termed the "induced-matter interpretation" of Kaluza-Klein theory, and eq. (55) is said to define the energy-momentum tensor of induced matter.

This tensor satisfies the appropriate requirements: it is symmetric (the first term is a second derivative, while the others are all explicitly symmetric), and reduces to the expected limit when the cylinder condition is re-applied (ie., when all derivatives ∂_4 with respect to the fifth dimension are dropped). In this case, the scalar wave equation (52) becomes just the Klein-Gordon equation for a massless scalar field:

$$\Box\phi = 0 \quad ; \tag{56}$$

and the contracted energy momentum tensor of the induced matter vanishes:

$$T = g^{\alpha\beta}T_{\alpha\beta} = 0 \quad , \tag{57}$$

which implies a radiationlike equation of state ($p = \rho/3$) for the induced matter, in agreement with earlier work [220] based on the cylinder condition. The induced matter in this case consists of photons, the gauge bosons of electromagnetism - exactly the same result obtained by Kaluza. This is the only kind of matter one can obtain in the induced-matter interpretation as long as the cylinder condition is in place. To extend Kaluza's approach to other kinds of matter, it is necessary to do one of two things: (1) go to higher dimensions and add an explicit energy-momentum tensor (or other terms) to the higher-dimensional vacuum field equations (compactified theories in practice involve both these things); or (2) loosen the restriction of cylindricity. In noncompactified theory, which takes the latter course, it turns out that matter described by $T_{\alpha\beta}$ - even in five dimensions - is already general enough to describe many physical systems, including in particular those connected with cosmology and the classical tests of general relativity.

The interpretation of eqs. (50) and (52) - the $\alpha 4$- and 44- components of the five-dimensional field equations ($\hat{R}_{AB} = 0$) - is not as straightforward as that of eq. (49). The relative simplicity of the conservation equation (50) suggests that there is a deeper physical significance to the four-tensor P^{β}_{α}, whose fully covariant form is $P_{\alpha\beta} \equiv (\partial_4 g_{\alpha\beta} - g_{\alpha\beta}g^{\gamma\delta}\partial_4 g_{\gamma\delta})/(2\sqrt{\hat{g}_{44}})$. It may be related to more familiar conserved physical quantities, or to the Bianchi identities [20].

Alternatively, it has been conjectured [177] that, as the $\alpha\beta$-components of the field equations link geometry with the *macroscopic* properties of matter, so the $\alpha 4$- and 44-components might describe their *microscopic* ones. In particular, if one makes the tentative identification:

$$P_{\alpha\beta} = k(m_i v_\alpha v_\beta + m_g g_{\alpha\beta}) \quad , \tag{58}$$

where k is a constant, m_i and m_g are the (suitably defined) inertial and gravitational mass of a particle in the induced-matter fluid, and $v^\alpha \equiv dx^\alpha/ds$ is its four-velocity, then the conservation equation (50) turns out to be the *four-dimensional geodesic equation* (for one class of metrics at least). This is interesting, since equations of motion are usually quite distinct from the field equations. Similarly, using appropriate definitions of particle mass m, one can identify the scalar wave equation (52) with the simplest possible relativistic quantum wave equation, namely the *Klein-Gordon equation*:

$$\Box\phi = m^2\phi \quad . \tag{59}$$

The relevant expression for particle mass turns out to depend explicitly on the components of the metric, which means that this variant of noncompactified Kaluza-Klein theory is a realization of Mach's Principle [251], [252]. These are interesting results, but speculative ones, and we do not discuss them further here. Some other Machian aspects of noncompactified theories have been explored in [25], [150], [178], [179], [184].

6.4 The Spherically-Symmetric Case

To appreciate what the induced-matter energy-momentum tensor (55) means physically, one has to supply a five-dimensional metric $\hat{g}_{AB}$ - preferably one specific enough to simplify the mathematics but general enough to be broadly applicable, eg., to both cosmological and one-body problems. We begin here with the *general spherically-symmetric* five-dimensional line element:

$$d\hat{s}^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - R^2(d\theta^2 + sin^2\theta d\phi^2) + \varepsilon e^{\mu}d\psi^2 \quad , \tag{60}$$

where ε serves the same function as before, t, r, θ and ϕ have their usual meanings, ψ is the fifth coordinate, and ν, λ, R and μ are, for now, arbitrary functions of r, t and ψ. Denoting derivatives with respect to t by overdots ($\dot{}$), derivatives with respect to r by primes ($'$), and derivatives with respect to ψ by star superscripts (*), one finds [21] that the energy-momentum tensor (55) of induced matter has the following nonzero components:

$$\begin{aligned}
8\pi G T^0_0 &= -e^{-\nu}\left(\frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} - \frac{\lambda'\mu'}{4} + \frac{\mu''}{2} + \frac{\mu'^2}{4}\right) \\
&\quad -\varepsilon e^{-\mu}\left(\frac{\lambda^{**}}{2} + \frac{\lambda^{*2}}{4} - \frac{\mu^*\lambda^*}{4} + \frac{R^*\lambda^*}{R} - \frac{R^*\mu^*}{R} + \frac{R^{*2}}{R^2} + \frac{2R^{**}}{R}\right) \\
8\pi G T^1_0 &= -e^{-\lambda}\left(\frac{\dot{\mu}'}{2} + \frac{\dot{\mu}\mu'}{4} - \frac{\nu'\dot{\mu}}{4} - \frac{\dot{\lambda}\mu'}{4}\right) \\
8\pi G T^1_1 &= -e^{-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} + \frac{\nu'\mu'}{4}\right) \\
&\quad -\varepsilon e^{-\mu}\left(\frac{R^{*2}}{R^2} + \frac{2R^{**}}{R} + \frac{R^*\nu^*}{R} - \frac{R^*\mu^*}{R} + \frac{\nu^{**}}{2} + \frac{\nu^{*2}}{4} - \frac{\mu^*\nu^*}{4}\right) \\
8\pi G T^2_2 &= -e^{-\nu}\left(\frac{\dot{R}\dot{\mu}}{2R} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4}\right) + e^{-\lambda}\left(\frac{R'\mu'}{2R} + \frac{\mu''}{2}\right. \\
&\quad \left. + \frac{\mu'^2}{4} - \frac{\lambda'\mu'}{4} + \frac{\nu'\mu'}{4}\right) - \varepsilon e^{-\mu}\left(\frac{R^{**}}{R} + \frac{R^*\nu^*}{2R} + \frac{R^*\lambda^*}{2R} - \frac{R^*\mu^*}{2R}\right. \\
&\quad \left. + \frac{\nu^{**}}{2} + \frac{\nu^{*2}}{4} + \frac{\lambda^{**}}{2} + \frac{\lambda^{*2}}{4} + \frac{\nu^*\lambda^*}{4} - \frac{\nu^*\mu^*}{4} - \frac{\mu^*\lambda^*}{4}\right) \\
T^3_3 &= T^2_2 \quad .
\end{aligned} \tag{61}$$

If one then assumes that this induced matter takes the form of a perfect fluid:

$$T^{\alpha}_{\beta} = (\rho + p)u^{\alpha}u_{\beta} - p\delta^{\alpha}_{\beta} \quad , \tag{62}$$

where u^{α} is the four-velocity of the fluid elements, then the density ρ and pressure p can be readily identified [21] from the relations $\rho = T^0_0 + T^1_1 - T^2_2$ and $p = -T^2_2$. Inserting the expressions (61), one obtains:

$$\begin{aligned}
8\pi G\rho &= \frac{3}{2}\left(\frac{e^{-\lambda}\mu'R'}{R} - \frac{e^{-\nu}\dot{\mu}\dot{R}}{R}\right) \\
&\quad +\varepsilon e^{-\mu}\left(\frac{\nu^*\lambda^*}{4} + \frac{3R^*\mu^*}{2R} - \frac{R^*\lambda^*}{2R} - \frac{R^*\nu^*}{2R} - \frac{2R^{*2}}{R^2} - \frac{3R^{**}}{R}\right) \quad ,
\end{aligned}$$

$$
\begin{aligned}
8\pi G p &= \frac{1}{2}\left(\frac{e^{-\lambda}\mu' R'}{R} - \frac{e^{-\nu}\dot{\mu}\dot{R}}{R}\right) \\
&\quad + \varepsilon e^{-\mu}\left(\frac{\nu^*\lambda^*}{4} + \frac{R^*\mu^*}{2R} + \frac{R^*\lambda^*}{2R} + \frac{R^*\nu^*}{2R} - \frac{R^{**}}{R}\right) \quad . \qquad (63)
\end{aligned}
$$

It is immediately apparent that under the restriction of cylindricity (all starred quantities vanish), one can obtain only *radiation* ($p = \rho/3$) from Kaluza's mechanism, as noted already.

With the relaxation of this condition, by contrast, one obtains a very *general* equation of state. For example, one can split the density and pressure into four components ($\rho = \rho_r + \rho_d + \rho_v + \rho_s$ and $p = p_r + p_d + p_v + p_s$), where the radiation component obeys $p_r = \rho_r/3$, the dust-like component obeys $p_d = 0$, the vacuum component obeys $p_v = -\rho_v$, and the stiff component obeys $p_s = \rho_s$. One then finds from eqs. (63) that:

$$
\begin{aligned}
\rho_r &= \frac{3}{16\pi G}\left(\frac{e^{-\lambda}\mu' R'}{R} - \frac{e^{-\nu}\dot{\mu}\dot{R}}{R}\right) + \frac{3\varepsilon e^{-\mu}}{8\pi G}\left(\frac{R^*\nu^*}{2R} - \frac{R^{**}}{R}\right) \quad , \\
\rho_d &= -\frac{\varepsilon e^{-\mu} R^{*2}}{4\pi G R^2} \quad , \\
\rho_v &= -\frac{\varepsilon e^{-\mu} R^*}{16\pi G}\left(\nu^* + \lambda^*\right) \quad , \\
\rho_s &= \frac{\varepsilon e^{-\mu}\nu^*\lambda^*}{32\pi G} \quad . \qquad (64)
\end{aligned}
$$

From the first of these equations, it follows that in a radiationlike universe whose metric coefficients depend only on time, the fifth dimension must contract with time ($\dot{\mu} < 0$) if one is to have spatial expansion ($\dot{R} > 0$) and positive density ($\rho_r > 0$). Mechanisms of this sort have been used in compactified Kaluza-Klein cosmology to pump entropy into the four-dimensional universe, solving the horizon and flatness problems [221]; or indeed to explain why the fifth dimension is compact in the first place [222] (see §7.1). In the noncompactified approach, they no longer have to be assumed *a priori*, but can be seen to be *required by the field equations*. From the second of the above equations, meanwhile, it follows that a dustlike universe must have a spacelike fifth dimension ($\varepsilon = -1$ in our convention) in order for its density to be positive ($\rho_d > 0$). This agrees with the causality argument (§6.1).

6.5 The Isotropic and Homogeneous Case

One can go farther by making additional assumptions about the metric. Suppose the line element (60) is rewritten in spatially isotropic form:

$$
d\hat{s}^2 = e^{\nu}dt^2 - e^{\omega}(dr^2 + r^2 d\Omega^2) + \varepsilon e^{\mu}d\psi^2 \quad , \qquad (65)
$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. If one assumes that ν, ω and μ are *separable* functions of the variables t, r and ψ, one can obtain specialized solutions to the field equations (2) whose properties of matter, as specified by the energy-momentum tensor (61), agree very closely with those expected from four-dimensional theory.

Consider first the case of dependence on t only. The metric (65) is then just a five-dimensional generalization of a flat homogeneous and isotropic Friedmann-Robertson-Walker (FRW) cosmology. But in the context of noncompactified Kaluza-Klein theory, one ought also to allow dependence on the

extra coordinate ψ. So the *general* flat five-dimensional cosmological metric, assuming separability, should have:

$$e^{\nu} \equiv T^2(t)X^2(\psi) \quad , \quad e^{\omega} \equiv U^2(t)Y^2(\psi) \quad , \quad e^{\mu} \equiv V^2(t)Z^2(\psi) \quad . \tag{66}$$

Ponce de Leon [145] was the first to investigate solutions of the vacuum Einstein equations (2) with this form. Of his eight solutions, one is of special interest because it reduces on hypersurfaces $\psi =$ constant to the spatially flat *four-dimensional* FRW metric. This solution has $\varepsilon = -1$ and:

$$\begin{aligned} T(t) = constant \quad &, \quad X(\psi) \propto \psi \quad , \\ U(t) \propto t^{1/\alpha} \quad &, \quad Y(\psi) \propto \psi^{1/(1-\alpha)} \quad , \\ V(t) \propto t \quad &, \quad Z(\psi) = constant \quad , \end{aligned} \tag{67}$$

and can be written in the form:

$$d\hat{s}^2 = \psi^2 dt^2 - t^{2/\alpha}\psi^{2/(1-\alpha)}(dx^2 + dy^2 + dz^2) - \frac{\alpha^2}{(1-\alpha)^2}t^2 d\psi^2 \quad , \tag{68}$$

where $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2$ are the usual rectangular coordinates, and α is a free parameter of the theory [168]. Because this solution reduces on spacetime sections ($d\psi = 0$) to the familiar $k = 0$ FRW metric:

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) \quad , \tag{69}$$

it can properly be called the *generalization of the flat FRW cosmological metric to five dimensions*.

Assuming that cosmological matter behaves like a perfect fluid, one obtains from eqs. (63) the following expressions for density and pressure [167]:

$$\rho = \frac{3}{8\pi G\alpha^2\psi^2 t^2} \quad , \quad p = \left(\frac{2\alpha}{3} - 1\right)\rho \quad . \tag{70}$$

These are consistent with a wide variety of equations of state: a radiation-dominated universe, for example, if $\alpha = 2$; a dust-filled one if $\alpha = 3/2$; or an inflationary one if $0 < \alpha < 1$. Physical properties of cosmologies based on the metric (68) have been explored in [167]-[171], and its implications for the equations of motion (eg., of galaxies) are known [172]. Generalizations to $k \neq 0$ cosmologies [173], [174] and extended (eg., Gauss-Bonnet) theories of gravity [175], [176] have been made, and a connection to Mach's principle [25], [177]-[179] has been identified. These and related issues are reviewed in §7.

One other of Ponce de Leon's homogeneous and isotropic solutions [145] deserves mention. It has:

$$\begin{aligned} T(t) = constant \quad &, \quad X(\psi) \propto \psi \quad , \\ U(t) \propto \exp\left(\sqrt{\Lambda/3}t\right) \quad &, \quad Y(\psi) \propto \psi \quad , \\ V(t) = constant \quad &, \quad Z(\psi) = constant \quad , \end{aligned} \tag{71}$$

and looks like:

$$d\hat{s}^2 = \psi^2 dt^2 - \psi^2 e^{2\sqrt{\Lambda/3}t}(dx^2 + dy^2 + dz^2) - d\psi^2 \quad . \tag{72}$$

This reduces on spacetime hypersurfaces (ψ = constant) to the de *Sitter* metric, and $\Lambda \equiv 3/\psi^2$ is a *cosmological constant* induced in four-dimensional spacetime by the existence of the fifth coordinate ψ. The equation of state of the "matter" induced in four dimensions is that of the classical de Sitter vacuum, $p = -\rho$, with $\rho = \Lambda/(8\pi G)$.

Billyard & Wesson [171] have considered generalizations of this solution:

$$d\hat{s}^2 = \psi^2 dt^2 - \psi^2 e^{i\omega t}(e^{ik_1 x}dx^2 + e^{ik_2 y}dy^2 + e^{ik_3 z}dz^2) + \ell^2 d\psi^2 \quad , \tag{73}$$

where ω is a frequency, k_i a wave vector, and ℓ measures the size of the extra dimension. The induced-matter equation of state is again $p = -\rho$, but now with $\rho = -3\omega^2/(32\pi G\psi^2)$. The field equations (2) turn out to require $\ell^2 = 4/\omega^2$ so the vacuum has positive energy density if the fifth dimension is spacelike. The metric coefficients of ordinary three-space exhibit wave-like behaviour, but the associated medium is an unperturbed de Sitter vacuum - so this solution describes what might be termed "vacuum waves" in Kaluza-Klein theory. (They are not gravitational waves of the conventional sort because three-space is spherically-symmetric.) One might apply this to the inflationary universe scenario; imagining, for example, that ω starts out with real values (corresponding to a vacuum-dominated universe with oscillating three-space coefficients) but later takes on *imaginary* values (for which the universe enters an expanding de Sitter phase of the usual kind). On this interpretation, the big bang occurs as a (presumably quantum-induced) phase change - as has previously been suggested elsewhere on other grounds [223], [224].

6.6 The Static Case

The metric (65) reduces to another well-known form when the coefficients ν, ω and μ depend only on the radial coordinate r. This is just a five-dimensional generalization of the one-body or Schwarzschild metric, and has been variously interpreted in the literature as describing a magnetic monopole [225], "black hole" [227], and *soliton* [226] (see §8.1 for discussion). Again, however, from a noncompactified point of view there is no *a priori* reason to suppress dependence on ψ, so a *general* static spherically-symmetric metric, assuming separability, should have:

$$e^{\nu} \equiv A^2(r)D^2(\psi) \quad , \quad e^{\omega} \equiv B^2(r)E^2(\psi) \quad , \quad e^{\mu} \equiv C^2(r)F^2(\psi) \quad . \tag{74}$$

Ponce de Leon & Wesson [21] searched for two-parameter solutions of the five-dimensional field equations (2) with this form and found only four. (Liu & Wesson [193], [194] have recently obtained a three-parameter generalization of this class). The most useful is the one which contains the ordinary *four-dimensional* Schwarzschild solution as a limiting case. This is the solution with D, E and F constant (= 1 without loss of generality), and is thus identical to the soliton metric just mentioned. The coefficients A, B and C are:

$$\begin{aligned} A(r) &= \left(\frac{ar-1}{ar+1}\right)^{\epsilon k} \quad , \quad B(r) = \frac{1}{a^2 r^2}\frac{(ar+1)^{\epsilon(k-1)+1}}{(ar-1)^{\epsilon(k-1)-1}} \quad , \\ C(r) &= \left(\frac{ar+1}{ar-1}\right)^{\epsilon} \quad , \end{aligned} \tag{75}$$

where a is a constant related to the mass of the central body, and ϵ and k are other parameters (in the notation of [227]). Only one of these is strictly a free parameter, as they are related by a consistency relation:

$$\epsilon^2(k^2 - k + 1) = 1 \quad . \tag{76}$$

Written out explicitly, the metric is:

$$d\hat{s}^2 = A^2(r)dt^2 - B^2(r)(dr^2 + r^2 d\Omega^2) - C^2(r)d\psi^2 \quad , \tag{77}$$

where we have assumed a spacelike fifth coordinate ($\varepsilon = -1$) in agreement with other work. In the limits $\epsilon \to 0$, $k \to \infty$, and $\epsilon k \to 1$ (where $a \equiv 2/GM_*$ and M_* is the mass of the central body), this metric reduces on spacetime sections $d\psi = 0$ to the familiar Schwarzschild metric (in isotropic coordinates):

$$ds^2 = \left(\frac{1 - GM_*/2r}{1 + GM_*/2r}\right)^2 dt^2 - \left(1 + \frac{GM_*}{2r}\right)^4 (dr^2 + r^2 d\Omega^2) \quad . \tag{78}$$

It is therefore properly called the *generalization of the Schwarzschild metric to five dimensions*. Elsewhere in §6 we will refer to the above values of and k as the "Schwarzschild limit" of the theory.

Assuming as usual that the induced matter takes the form of a perfect fluid, eqs. (63) give for both solutions the following density and pressure [180]:

$$\rho = \frac{\epsilon^2 k a^6 r^4}{2\pi G(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\epsilon(k-1)} \quad , \quad p = \frac{\rho}{3} \quad . \tag{79}$$

The soliton metric (77) thus describes a central mass surrounded by an inhomogeneous cloud of radiation-like matter whose density goes as $\sim 1/a^2 r^4$ at large values of r. (The Schwarzschild limit defined above is the special case where the density and pressure of the cloud are zero; in this case $p = 0 = -\rho$ which is the usual vacuum solution, with its attendant classical tests of general relativity.) The $\epsilon^2 k$-term indicates that this combination of the two (related) parameters ϵ and k may characterize the soliton's energy density [181]. (This is somewhat different from the traditional interpretation, in which these parameters are related to its "scalar charge" [215], [228].) The equation of state (79) obtained in the induced-matter interpretation differs from the one found by Davidson & Owen [227], who used an approach based on Kac-Moody symmetries [59] and concluded that $p = -\rho/3$. Density shows the same r-dependence at large distances in both approaches, however, and goes to zero in the same Schwarzschild limit. Both solutions are invariant under $a \to -a$, $\epsilon \to -\epsilon$, and require $k > 0$ for positive density. One can define a *pressure three-tensor* p^a_b in the induced-matter interpretation, using the ab-components of the vacuum field equations (2), and this yields a result [180] very similar to the series expression obtained by Davidson & Owen. If one then takes $p \equiv p^a_a/3$ (as in [227]), one gets back exactly the result in eq. (79).

In Cartesian spatial coordinates the pressure tensor in general contains off-diagonal components, which implies that the matter making up the soliton is a sum of both a material (perfect) fluid and a free electromagnetic field [180]. The terms "density" and "pressure" therefore have to be treated with caution. Solitonic matter, in fact, is best be described as a *relativistic fluid with anisotropic pressure* [181]. Anisotropic spherically-symmetric fluids have an energy-momentum tensor given by:

$$T_{\alpha\beta} = (\rho + p_\perp)u_\alpha u_\beta + (p_\parallel - p_\perp)\chi_\alpha \chi_\beta + p_\perp g_{\alpha\beta} \quad , \tag{80}$$

where χ_α is a unit spacelike vector orthogonal to u_α, and $p_\parallel$, $p_\perp$ refer to pressure parallel and perpendicular to the radial direction. Assuming that the induced matter (61) takes this form rather than that of the perfect fluid (62); and choosing $u^\alpha = (u^0, 0, 0, 0)$, $\chi^\alpha = (0, \chi^1, 0, 0)$, one finds:

$$p_\parallel \;=\; \left[1 - \left(\frac{a^2 r^2 - 2\epsilon(k-1)ar + 1}{\epsilon ar}\right)\right]\rho \quad ,$$

$$p_\perp = \left(\frac{a^2r^2 - 2\epsilon(k-1)ar + 1}{2\epsilon ar}\right)\rho \quad , \tag{81}$$

with ρ exactly as in eq. (79). These expressions satisfy $\rho = p_\parallel + 2p_\perp$, confirming that the fluid has the nature of radiation. The physical properties of solitons based on the metric (77) have been studied by several authors [180]-[183], [225]-[228]. Their implications for astrophysics [184], [185], the classical tests of general relativity [187]-[189], and the equivalence principle [190] have been explored, and the class has been extended to time-dependent [191], [192] and charged solutions [193], [194]. These and related issues are reviewed in §8.

We mention for completeness the other three static solutions of the form (74) obtained by Ponce de Leon & Wesson [21]. Two of them have $A(r)$, $B(r)$ and $C(r)$ exactly as in eqs. (75), but have $F(\psi)$ an *arbitrary* function of ψ, with $D^*(\psi) \propto F(\psi)$ and $E(\psi) =$ constant in the first case, and $D(\psi) \propto E^{-1}(\psi)$ and $E^*(\psi) \propto F(\psi)/E(\psi)$ in the second one. The density and pressure for both these solutions is exactly as in eqs. (79) above, except for an added factor of $E^2(\psi)$ in the denominators. This is physically innocuous in the first case ($E(\psi) =$ constant) but means in the second one that these attributes of the radiation cloud depend on the extra coordinate. The ψ-dependent components of the field equations place an extra constraint on these both these solutions, restricting the allowed values of the parameters ϵ and k. The final solution is more interesting, and can be written in the form:

$$d\hat{s}^2 = \frac{\psi^2}{\psi^2 + \ell}dt^2 - \frac{(\psi^2 + \ell)^2}{(ar^2 + b)}(dr^2 + r^2 d\Omega^2) + \varepsilon d\psi^2 \quad , \tag{82}$$

where a is related to the mass of the central object as before, $b = -\varepsilon/4a$, and ℓ is one other independent constant of the system. Although this solution was found by assuming separability in r and ψ, it also satisfies the field equations (2) when a and b are arbitrary functions of ψ. This is intriguing, as it hints at a relationship between the mass of the central object and the fifth coordinate. Another interesting feature of the metric (82) is its induced-matter equation of state, which - unlike that of the other soliton solutions found so far - is not radiationlike, but turns out to be the one discussed by Davidson & Owen [227]: $p = -\rho/3$. This is an unusual form of matter, but has been considered previously in several other contexts [229]-[232], largely because it describes matter that does not disturb other objects gravitationally (gravitational or Tolman-Whittaker mass is proportional to $3p + \rho$). Thus it might, for instance, be useful in reconciling the extremely high energy densities expected for quantum zero-point fields with the small value observed for Einstein's cosmological constant [231].

6.7 General Covariance in Higher Dimensions

We have reviewed a number of solutions to the spherically-symmetric vacuum field equations in five dimensions. In each case the five-dimensional geometry manifests itself in four dimensions as induced matter, with an associated equation of state. The equation of state, in fact, *follows from* the field equations in the induced-matter interpretation of Kaluza-Klein theory, rather than having to be supplied separately as in four-dimensional general relativity. In several cases, the physical form of the metric on spacetime hypersurfaces $d\psi =$ constant, or the equation of state for the induced matter, is such as to make the solutions useful for testing the predictions of noncompactified theory. The theory is so far not in conflict with any experimental data (see §7 and §8).

However, it is important to keep in mind that physical quantities such as the scalars ρ and p, which are designed to be invariant with respect to *four-dimensional* coordinate changes $x^\alpha \to \tilde{x}^\alpha$ cannot in general stay that way in noncompactified Kaluza-Klein theory, which is invariant with respect to

five-dimensional ones $x^A \to \tilde{x}^A$. Any quantities - even those normally thought of as conserved - are vulnerable if they depend on the fifth coordinate x^4.

What this means in practice is that density, pressure, and the equation of state in the induced-matter interpretation are to some extent dependent on the coordinates in which one chooses to express them. A search for the correct solution to a (four-dimensional) physical problem is also a search for the appropriate system of (five-dimensional) *coordinates*. This can perhaps best be illustrated with a series of simple x^4-dependent coordinate transformations [23], beginning with five-dimensional Minkowski space:

$$d\hat{s}^2 = dt^2 - dr^2 - r^2 d\Omega^2 - d\psi^2 \quad . \tag{83}$$

Spacetime sections of this metric are of course four-dimensional Minkowski spaces. If one transforms to primed coordinates:

$$t' = t \quad , \quad r' = \frac{r}{\psi}\left(1 + \frac{r^2}{\psi^2}\right)^{-1/2} \quad , \quad \psi' = \psi\left(1 + \frac{r^2}{\psi^2}\right)^{1/2} \quad , \tag{84}$$

this metric becomes:

$$d\hat{s}'^2 = dt'^2 - \psi'^2\left(\frac{dr'^2}{1 - r'^2} + r'^2 d\Omega^2\right) - d\psi'^2 \quad . \tag{85}$$

Spacetime sections ($\psi' =$ constant) of the new primed metric are *static Einstein cosmologies*; ie., four-dimensional FRW metrics:

$$ds'^2 = dt'^2 - R^2(t')\left(\frac{dr'^2}{1 - kr'^2} + r'^2 d\Omega^2\right) \quad , \tag{86}$$

with $k = +1$ and a *constant* scale factor $R(t') = \psi'$. One can obtain from the Friedmann equations the value of Einstein's cosmological constant Λ, and (assuming a perfect fluid) expressions for the density and pressure of matter:

$$\Lambda = \frac{1}{\psi'^2} \quad , \quad \rho_m = \frac{1}{4\pi G \psi'^2} \quad , \quad \rho_m = 0 \quad . \tag{87}$$

The cosmological constant represents a vacuum energy density $\rho_v = \Lambda/(8\pi G)$ with associated pressure $p_v = -\rho_v$. So altogether one has:

$$\rho = \rho_m + \rho_v = \frac{3}{8\pi G \psi'^2} \quad , \quad p = \rho_m + \rho_v - \frac{\rho}{3} \quad . \tag{88}$$

The effective equation of state in the four-dimensional spacetime sections of the primed metric (85) is thus that of *non-gravitating matter* of the kind discussed in §6.6. (The same result could have been obtained by plugging the metric directly into eqs. (63) for induced-matter density and pressure.) A second coordinate transformation to double-primed coordinates:

$$t' = \psi'' \sinh t'' \quad , \quad r' = r'' \quad , \quad \psi' = \psi'' \cosh t'' \quad , \tag{89}$$

puts the metric into the new form:

$$d\hat{s}''^2 = \psi''^2 dt''^2 - \psi''^2 \cosh^2 t'' \left(\frac{dr''^2}{1 - r''^2} + r''^2 d\Omega^2 \right) - d\psi''^2 \quad . \tag{90}$$

Spacetime sections of this double-primed metric are *expanding FRW cosmologies*, with $k = +1$ and $R(t'') = \psi'' \cosh t''$. The density and pressure of the accompanying perfect fluid, as obtained from eqs. (63), are:

$$\rho = \frac{3}{8\pi G \psi''^2} \quad , \quad p = -\rho \quad , \tag{91}$$

so that the effective equation of state is that of a *pure vacuum*.

Of the metrics (83), (85) and (90), which one is the best choice for a description of the real universe? None, of course, since none of them admits spacetime sections with realistic four-dimensional properties. A metric which *is* adequate to this task is the cosmological one (68). It can be obtained from Minkowski space (83) by transforming from t, r, ψ to $\tilde{t}$, $\tilde{r}$, $\tilde{\psi}$ via:

$$\begin{aligned} t &= \frac{\alpha}{2}\left(1 + \frac{\tilde{r}^2}{\alpha^2}\right)\tilde{t}^{1/\alpha}\tilde{\psi}^{1/(1-\alpha)} - \frac{\alpha}{2(1-2\alpha)}\left[\tilde{t}^{-1}\tilde{\psi}^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha} \quad , \\ r &= \tilde{r}\tilde{t}^{1/\alpha}\tilde{\psi}^{1/(1-\alpha)} \quad , \\ \psi &= \frac{\alpha}{2}\left(1 - \frac{\tilde{r}^2}{\alpha^2}\right)\tilde{t}^{1/\alpha}\tilde{\psi}^{1/(1-\alpha)} + \frac{\alpha}{2(1-2\alpha)}\left[\tilde{t}^{-1}\tilde{\psi}^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha} \quad , \end{aligned} \tag{92}$$

and dropping the tildes [25]. The cosmological metric, as we have seen (§6.5), gives back good models for the early (radiation) and late (dust) universe on spacetime sections $\psi =$ constant if α is chosen appropriately.

The point of this exercise is that *all four of the metrics (68), (83), (85) and (90) are flat in five dimensions*, although they would be perceived very differently by four-dimensional observers (as evinced by their expansion factors and equations of state). The reason for these differences is the ψ-dependence of the coordinate transformations, and the fact that the theory is covariant with respect to *five*-, not four-dimensional coordinates. To properly describe a given four-dimensional problem in noncompactified theory, one needs to choose five-dimensional coordinates judiciously. This is not a reflection of some fundamental ambiguity in the theory, but is rather forced on us as long as we insist on retaining four-dimensional concepts like density and pressure in a five-dimensional theory (see also §7.6).

6.8 Other Exact Solutions

Similar remarks apply to astrophysical situations. One has to choose five-dimensional coordinates appropriate to each problem, if one wants to couch the results in terms of familiar four-dimensional quantities. There is thus a rich field here for future inquiry. The one-body metric which has received most attention so far is that of the soliton (77), which contains the four-dimensional Schwarzschild solution on spacetime sections. As we saw in §6.6, however, the induced matter associated with this metric is necessarily radiationlike (except in the Schwarzschild limit), and its density falls off with distance rather steeply. To describe bodies with *different* properties, one must find new static spherically-symmetric solutions of the field equations. This is possible in Kaluza-Klein theory because Birkhoff's theorem (which guarantees the uniqueness of the Schwarzschild solution in four dimensions) no longer holds in higher dimensions [22], [184], [214], [226].

One such solution has recently been found by Billyard & Wesson [186]. It is actually a modification of the cosmological metric (68):

$$\begin{aligned} d\hat{s}^2 &= \left(\frac{r}{r_0}\right)^{2(\alpha+1)} \psi^{2(\alpha+3)/\alpha} dt^2 - (3-\alpha^2)\psi^2 dr^2 - \psi^2 r^2 d\Omega^2 \\ &\quad +3(3\alpha^{-2}-1)r^2 d\psi^2 \quad , \end{aligned} \tag{93}$$

where r_0 is a constant and α is a parameter related to the properties of matter. On spacetime hypersurfaces $d\psi = 0$ this metric is very similar to a four-dimensional one originally used to describe inhomogeneous spheres of matter in static isothermal equilibrium [233]. With the aid of eqs. (63), one finds that the associated induced matter has:

$$\rho = \frac{(2-\alpha^2)}{8\pi G(3-\alpha^2)\psi^2 r^2} \quad , \quad p = \left(\frac{\alpha^2+2\alpha+2/3}{2-\alpha^2}\right)\rho \quad . \tag{94}$$

In addition, one can use the standard (Tolman-Whittaker) definition [234] of the gravitational mass of a volume of fluid to obtain:

$$M_g(r) = \frac{(1+\alpha)}{G\sqrt{3-\alpha^2}} \psi^{2+3/\alpha} \left(\frac{r}{r_0}\right)^{2+\alpha} r_0 \quad . \tag{95}$$

The object described by this metric has positive density for $\alpha^2 \leq 2$, and positive mass (assuming $\alpha \neq 0$) for $\alpha \geq -1$. So altogether one has a nonzero α between -1 and $\sqrt{2}$, which allows for equations of state (94) anywhere in the range $-\rho/3 \leq p \leq \rho$. These are potentially relevant to a wide variety of astrophysical problems. But the fact that ρ and p are both proportional to r^{-2}, rather than $\sim r^{-4}$ as for solitons, indicates that eq. (93) may be especially useful for modelling phenomena such as galaxies, or clusters of them [235]-[237]. To go further one needs to rederive the classical tests of general relativity for this metric, just as has been done for the soliton one (see §8). Some work has been done in this direction in [186].

6.9 The Equations of Motion

Like the higher-dimensional field equations, the higher-dimensional equations of motion are also modified when dependence is allowed on extra coordinates. In this section, in order to explicitly include electromagnetic effects, we no longer restrict our choice of coordinates to those in which $A_\alpha = 0$. The metric $\hat{g}_{AB}$ is given by eq. (5), with the addition of the ε-factor to allow for timelike, as well as spacelike x^4. We then obtain the equations of motion by minimizing the five-dimensional interval $d\hat{s}^2 = \hat{g}_{AB} dx^A dx^B$. This results in a five-dimensional version of the geodesic equation [172]:

$$\frac{d^2 x^A}{d\hat{s}^2} + \hat{\Gamma}^A_{BC} \frac{dx^B}{d\hat{s}} \frac{dx^C}{d\hat{s}} = 0 \quad , \tag{96}$$

with the five-dimensional Christoffel symbol defined as in eq. (4). The $A = 4$ component of eq. (96) can be shown [172] to take the form:

$$\frac{d\mathcal{B}}{d\hat{s}} = \frac{1}{2} \frac{\partial \hat{g}_{CD}}{\partial x^4} \frac{dx^C}{d\hat{s}} \frac{dx^D}{d\hat{s}} \quad , \tag{97}$$

where $\mathcal{B}$ is a scalar function:

$$\mathcal{B} \equiv \varepsilon\phi^2\left(\frac{dx^4}{d\hat{s}} + \kappa A_\alpha \frac{dx^\alpha}{d\hat{s}}\right) \quad . \tag{98}$$

In the case where $\hat{g}_{AB}$ does not depend on x^4, $\mathcal{B}$ is a constant of the motion (since $d\mathcal{B}/d\hat{s} = 0$), but this is not generally so in noncompactified theory. The definition of $\mathcal{B}$, together with the form of the metric (5), allow us to express the five-dimensional interval in terms of the four-dimensional one via $d\hat{s} = (1 - \varepsilon\mathcal{B}^2/\phi^2)^{-1/2} ds$. Using this relation, the $A = \alpha$ components of eq. (96) can be shown [172] to take the form:

$$\begin{aligned}\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} &= \frac{\mathcal{B}}{\sqrt{1-\varepsilon\mathcal{B}^2/\phi^2}}\left[F^\mu_\nu \frac{dx^\nu}{ds} - \frac{\kappa A^\mu}{\mathcal{B}}\frac{d\mathcal{B}}{ds} - \kappa g^{\mu\lambda}\frac{\partial A_\lambda}{\partial x^4}\frac{dx^4}{ds}\right] \\ &\quad + \frac{\varepsilon\mathcal{B}^2}{(1-\varepsilon\mathcal{B}^2/\phi^2)\phi^3}\left[\nabla^\mu\phi + \left(\frac{\phi}{\mathcal{B}}\frac{d\mathcal{B}}{ds} - \frac{d\phi}{ds}\right)\frac{dx^\mu}{ds}\right] \\ &\quad - g^{\mu\lambda}\frac{\partial g_{\lambda\mu}}{\partial x^4}\frac{dx^\nu}{ds}\frac{dx^4}{ds} \quad .\end{aligned} \tag{99}$$

This is the fully general equation of motion in Kaluza-Klein theory, and for $\mu = 1, 2, 3$ shows how a test particle moves in ordinary space.

The left-hand side of eq. (99) is identical to that in Einstein's theory; the terms on the right-hand side are deviations from four-dimensional geodesic motion. In the case of no dependence on the extra coordinate x^4, the four terms in $d\mathcal{B}/ds$ and $\partial/\partial x^4$ all vanish and we correctly recover the same result obtained previously by those working in compactified Kaluza-Klein theory [238]-[240]. The terms in the first set of square brackets depend on a nonvanishing electromagnetic potential A_α, and the first of these can be recognized as the Lorentz force if the charge-to-mass ratio of the test particle is:

$$\frac{Q}{M} = \frac{\mathcal{B}}{\sqrt{1-\varepsilon\mathcal{B}^2/\phi^2}} \quad . \tag{100}$$

This relation, however, is only useful in the limit where the metric is independent of x^4, and its extra-dimensional part is flat [172]. In coordinate frames where this is not the case, one cannot readily identify quantities like mass or charge, which after all are four-dimensional concepts. The same caution applies to the "scalar charge-to-mass ratio" given by:

$$\frac{Q'}{M} = \frac{\varepsilon\mathcal{B}^2}{(1-\varepsilon\mathcal{B}^2/\phi^2)\phi^3} \quad , \tag{101}$$

which can be identified analogously to the electromagnetic one from the multiplicative factor in front of the *second* set of square brackets in eq. (99).

The 0-component of the geodesic equation (99), meanwhile, can be written [172] in a form analogous to eq. (97) above:

$$\frac{d\mathcal{C}}{ds} = \frac{1}{2}\frac{\partial\hat{g}_{CD}}{\partial x^0}\frac{dx^C}{ds}\frac{dx^D}{d\hat{s}} \quad , \tag{102}$$

where $\mathcal{C}$ is a new scalar function:

$$\mathcal{C} \equiv \sqrt{\frac{g_{00}}{1-v^2}\left(1 - \frac{\varepsilon\mathcal{B}^2}{\phi^2}\right) + \kappa\mathcal{B}A_0} \quad . \tag{103}$$

Here $v^2 = \lambda_{ab} v^a v^b$ is the square of the test particle's spatial 3-velocity $v^a = dx^a/(\sqrt{g_{00}}[dx^0 + (g_{0b}/g_{00})dx^b])$, with $\lambda_{\alpha\beta} \equiv g_{\alpha 0} g_{\beta 0}/g_{00} - g_{\alpha\beta}$ a suitable projector. If the metric $\hat{g}_{AB}$ were independent of time x^0 then $\mathcal{C}$ would be a constant of motion. Where this is not the case, as in cosmology, the geodesic equation (99) could in principle be applied to test noncompactified theory. We return to this question in §7.5. The possible physical significance of the quantity $\mathcal{C}$ is explored in more detail in [172].

6.10 Physical Meaning of the Fifth Coordinate

We have noted that the charge of a test particle can be readily identified in the limit as $\psi \equiv x^4 =$ constant. We have also found that a variety of realistic four-dimensional cosmological models and one-body metrics can be identified as constant-ψ hypersurfaces of flat five-dimensional Minkowski space. So far, then, it appears that useful coordinate systems can be specified by the condition $u^4 \equiv d\psi/ds = 0$. (This is perfectly legitimate from a mathematical point of view as the introduction of a fifth coordinate into general relativity means an extra degree of freedom that can always be used if one wishes to set a condition on u^4.) However, we have not much improved on Kaluza's cylinder condition unless we confront the question: are there any *physical* reasons why we should expect $d\psi/ds = 0$?

In answering this one is obliged to interpret ψ physically. We review here one such interpretation, which has been advanced by Wesson and his collaborators [23], [25], [135], [168]. Noncompactified theory in general (and elsewhere in this report, including the next two parts on experimental constraints) stands or falls quite independently of this additional work. The proposal we consider is that the fifth coordinate ψ might be related to *rest mass*. The coordinate frame picked out by $u^4 = 0$ is then just the one in which *particle rest masses are constant*. There are at least three independent pieces of evidence (besides the empirical fact that rest masses are conserved!) in support of this conjecture: (1) All of mechanics depends on base units of length, time and mass. So if the former two can be treated as coordinates, then maybe the last should also. Dimensionally, $x^4 = Gm/c^2$ allows us to treat the rest mass m of a particle as a length coordinate, in analogy with $x^0 = ct$. (2) Metrics which do not depend on x^4, like the soliton metric (77), can give rise only to induced matter composed of *photons*; while those which depend on x^4, like the cosmological metric (68), give back equations of state for fluids composed of *massive particles*. (3) The metrics $d\hat{s}^2 = dT^2 - d\sigma^2 - d\Psi^2$ and $d\hat{s}^2 = \psi^2 dt^2 - d\sigma^2 - t^2 d\psi^2$ are related by the coordinate transformations $T = t^2\psi^2/4 + \ln[(t/\psi)^{1/2}]$ and $\Psi = t^2\psi^2/4 - \ln[(t/\psi)^{1/2}]$. The former metric is flat, while the latter gives an action principle $\delta \int dt = 0$ for particles at rest in ordinary space ($d\sigma/ds = 0$), viewed on hypersurfaces $\psi =$ constant. This action principle is formally the same as that of particle physics if $\psi \to m$ in the local, low-velocity limit. (The same argument applies to cosmological metrics (68).) This view of the origin of mass is similar to that in some quantum field theories [241], where rest masses are generated spontaneously in a conformally invariant theory that includes a scalar dilaton field or Nambu-Goldstone boson in Minkowski space.

Several other, more philosophical reasons [23], [25] to consider the STM ("Space-Time-Matter") hypothesis that ψ might be related to m can perhaps be mentioned here: (4) A theory in which mass is placed on the same footing as space and time will be naturally *scale-invariant*, simply by virtue of being *coordinate-invariant* (because particle masses are a necessary part of any system of units, or "scales"). The idea that nature might be scale-invariant has been considered from time to time by such eminent thinkers as Dirac, Hoyle and others [23], [242]-[250]. (The STM approach is, however, otherwise quite distinct from these theories, not least in the fact that it predicts a variation in rest mass m rather than the dimension-transposing constant G.) (5) There is also a pleasing symmetry in the elevation of G to the same status as c: as the latter puts *distances* into temporal units, so the former is needed to do the same for *masses*. The actual conversion factors are $1/c$ and G/c^3 respectively, and this helps explain why any change in mass with time - a generic feature of

scale-invariant theories - has been so small as to have escaped detection so far: the latter factor is some 43 orders of magnitude smaller than the former, and the former is already tiny enough to have made special relativistic effects unnoticeable until the second half of this century. (6) Finally, we note that if x^4 is not restricted to be lengthlike (or timelike) in nature, then the extra part of the metric can have either sign without running afoul of closed timelike curves and causality problems (§6.1). We will not consider the STM theory further in this report, noting however that its observational implications have been studied over the years by Wesson [19], [136]-[139] and numerous others [140]-[148], [149]-[158], [159]-[163].

7 Cosmology

7.1 Compactified Kaluza-Klein Cosmology

Cosmological aspects of compactified Kaluza-Klein theory have received less attention than those related to particle physics [12]. Where they have been addressed [7], [13], much of the discussion has focused on the search for exact solutions of higher-dimensional general relativity (or extended gravity theories) which contain the familiar FRW universes on spacetime-like sections. This was first done in five dimensions (with no extra-dimensional matter) by Chodos & Detweiler [222] in 1980, and extended to ten- and eleven-dimensional supergravities (with appropriate higher-dimensional matter tensors) by Freund [253]. The key feature of these and subsequent models [254]-[260], [261]-[267] was that extra dimensions could (and in some cases necessarily *would*) shrink as the spatial ones expanded, thus lending support to the whole notion of compactification. (The possibility that compact subspaces could "bounce back" from a contracting phase was also investigated [268], [269].) This approach to explaining why the universe appears four-dimensional is sometimes referred to as "dynamical" or "cosmological" dimensional reduction. *Non*-compactified theory can exhibit the same behaviour, as noted in §6.4 and §8.10.

In more than five dimensions, compactification requires either explicit matter terms or modifications to the Einstein equations (§4.4). All kinds of matter have been invoked to induce cosmological compactification (usually in addition to that already required for spontaneous compactification; eg., as in supergravity [253], [256]-[258]). There are theories with dilaton fields [258], quantized five-dimensional scalar fields [259], a D-dimensional gas of non-interacting scalar particles [260], general higher-dimensional perfect fluids [256], [261]-[264], D-dimensional radiation [265], five-dimensional dust [266], and scalar fields in nonlinear sigma models [267]. Cosmological compactification mechanisms based on modifications of Einstein's theory of gravity are just as colorful, employing quadratic [270], [271], cubic [272], and even quartic terms [273] in the curvature, both generally and in special combinations known variously as Gauss-Bonnet terms [255], [265], [274], [275], Lanczos terms [276], Lovelock terms [273], [277], Euler-Poincaré densities [278], and dimensionally continued Euler forms [279], [280]. Even changes of metric signature [281] have been considered as instruments of compactification. An exhaustive survey and classification of generalized higher-dimensional vacuum cosmologies has recently been carried out by Coley [175].

An important step in Kaluza-Klein cosmology was the demonstration that shrinking extra dimensions could transfer entropy into the four-dimensional universe, providing a new way to solve the horizon and flatness problems [221], [282], although many (~ 40) extra dimensions were required [283]. Inflation itself has also been incorporated directly in compactified Kaluza-Klein theories [284], [285], and indeed "Kaluza-Klein inflation" has burgeoned into a sub-field of its own [286]. It is difficult to obtain in some supergravity [256] and most superstring [287] theories, and again requires in general that either additional matter terms [284] or higher-derivative corrections [285] be added to Einstein's theory. Examples of the former include higher-dimensional dust [288], scalar fields with conformal transformations of the metric [64], [289] or non-minimal couplings to the curvature [290], generalized

perfect fluids [291], [292], and others [293]. Examples of the latter include higher-derivative corrections to Einstein's equations [294], [295], Gauss-Bonnet terms [274], [281], and Euler forms [280]. Inflation has also been obtained with multiple compact subspaces [296] and explicit "chaotic inflaton" fields [297]. Other inflationary Kaluza-Klein cosmologies include versions of extended inflation [298] and STM theory [147]. An exciting recent development is the use of *COBE* measurements of microwave background anisotropy to put experimental limits - surprisingly restrictive ones in some cases - on inflationary Kaluza-Klein models [299]-[302].

Cosmological constraints on compactified Kaluza-Klein theories *apart* from those relating to inflation in the early universe have also received attention, beginning with Marciano's observation [303] that time-variation in the scale of extra dimensions would have important consequences for the fundamental constants of four-dimensional physics. Implications of the same phenomenon for primordial nucleosynthesis [304] and nuclear resonance levels in carbon and oxygen atoms [305] have also been discussed. If the extra dimensions are spatial in nature, these arguments imply that the present rate of change in their mean radius is less than about $10^{-19} yr^{-1}$. Another interesting idea is to use observations of *gravitational waves* to constrain Kaluza-Klein cosmologies; this however turns out to be impractical at the present time [306]. Other issues in compactified Kaluza-Klein cosmology include the possibility of excessive contributions to the global energy density from massive Fourier modes [13], [307] and solitons [13], [308] (see also §8.4), gravitational effects due to massless scalar components of the compactified higher-dimensional metric [309], and the stability of solutions with respect to classical perturbations [310], chaotic behaviour [311], and quantum effects [212], [268], [312]. Inhomogeneous Kaluza-Klein cosmologies have been considered in [313].

7.2 The Equation of State

Compactified Kaluza-Klein cosmology, as described above, is characterized by a profusion of competing expressions for the energy-momentum tensor $\hat{T}_{AB}$ in higher dimensions, reflecting the fact that there is no consensus on how to define "higher-dimensional matter." In noncompactified cosmology, by contrast, one avoids this ambiguity with the natural and economical assumption that $\hat{T}_{AB} \equiv 0$; that is, that the universe in higher dimensions is *empty*.

This cannot be done in compactified theory because the cylinder condition imposes uncomfortable restrictions on the resulting equation of state (and other properties of matter) in four dimensions. Consider as a simple example the uniform five-dimensional line element (65) with $\nu = 0$, $\omega = \ln t$, and $\mu = -\ln t$:

$$d\hat{s}^2 = dt^2 - t d\sigma^2 - t^{-1} d\psi^2 \quad , \tag{104}$$

where $d\sigma^2 = dx^2 + dy^2 + dz^2$ is shorthand for the spatial part of the metric. This would be an acceptable solution in compactified cosmology in that its constant-ψ sections are FRW, none of the metric coefficients depend on ψ, and the extra coordinate shrinks with time. Indeed its spatial part grows in exactly the same way as that of a four-dimensional FRW model with the radiation equation of state, $p = \rho/3$. And in fact, in the induced-matter interpretation, this metric literally *does* describe radiation. That is, putting eq. (104) into the five-dimensional vacuum field equations $\hat{G}_{AB} = 0$ gives back the four-dimensional ones $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, where $T_{\alpha\beta}$ is the energy-momentum tensor of a perfect fluid with:

$$\rho = \frac{3}{32\pi t^2} \quad , \quad p = \frac{\rho}{3} \quad , \tag{105}$$

as can be shown explicitly using eqs. (63). (Units are such that $G = c = 1$ throughout §7, except where otherwise noted.) These are the same expressions as those used to describe the radiation era

in (flat) four-dimensional cosmology [168]. In fact, unless five-dimensional matter is put in to begin with, this metric is incapable of manifesting itself as anything *but* electromagnetic radiation in four dimensions (§6.3).

In noncompactified cosmology, by contrast, one can describe the universe at *any* stage of its history without higher-dimensional matter (or modifications to the higher-dimensional field equations). As discussed in §6.5, the best metric for this purpose is the "cosmological metric" (68). Consider first the case $\alpha = 2$, which looks like:

$$d\hat{s}^2 = \psi^2 dt^2 - t\psi^{-2} d\sigma^2 - 4t^2 d\psi^2 \quad . \tag{106}$$

This line element again has FRW-like constant-ψ sections, and gives exactly the same expressions for induced density and pressure as eqs. (105) above, provided that the unphysical coordinate label t is replaced by the proper time ψt. So it again describes a radiation-dominated universe, or one filled with relativistic particles such as neutrinos. This time, however, the metric coefficients depend on ψ, and the fifth dimension *grows* with time. Solutions of this type tend to be discarded in particle physics, where the assumed lengthlike nature of the extra coordinates constrains them to be very small at the present time [55]. Here we make no *a priori* assumptions about the physical nature of extra dimensions. This allows us to obtain more general kinds of cosmological matter [167], [168]. In the case $\alpha = 3/2$, for instance, the same metric (68) reads:

$$d\hat{s}^2 = \psi^2 dt^2 - t^{4/3}\psi^{-4} d\sigma^2 - 9t^2 d\psi^2 \quad , \tag{107}$$

which, from eqs. (70), represents matter with induced density and pressure:

$$\rho = \frac{1}{6\pi(\psi t)^2} \quad , \quad p = 0 \quad , \tag{108}$$

and therefore describes a *dust-filled* universe. One can also model *inflation* (in flat FRW models) by choosing $0 < \alpha < 1$. Provided one is willing to tolerate a dependence on the extra coordinate, then, and a non-lengthlike interpretation of its physical nature, one can describe the universe at any stage of its history as a manifestation of pure geometry in five dimensions. In every case, the parameters ρ and p appear as products of an underlying geometric theory, and the equation of state manifests itself as a *consequence* of the field equations. This is more satisfying than the usual situation in (four-dimensional) cosmology, where pressure and density have merely phenomenological status and the equation of state must be put into the theory by hand.

7.3 Extension to k $\neq$ 0 Cosmologies

The cosmological metric (68), and the others mentioned in §6.5, are all five-dimensional generalizations of *spatially flat* four-dimensional FRW spacetimes. One could also consider a *curved* version of the homogeneous and isotropic line element (65):

$$d\hat{s}^2 = e^{\nu} dt^2 - e^{\omega}\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) - e^{\mu} d\psi^2 \quad . \tag{109}$$

McManus [174] has investigated solutions of the vacuum Einstein equations (2) with this form. Like Ponce de Leon [145], he assumed that ν, ω and μ were *separable* functions, as given by eq. (66), with $T(t) = Z(\psi) =$ constant. He found four solutions with $k \neq 0$, each associated with a well-defined induced-matter equation of state. We list his final results here. In the first solution, $X(\psi)$ and $Y(\psi)$ are constant as well as $Z(\psi)$, and the line element reads:

$$d\hat{s}^2 = dt^2 - (-kt^2 + \xi t + \eta)d\chi^2 - \frac{(kt - \xi/2)^2}{-kt^2 + \xi t + \eta}d\psi^2 \quad , \tag{110}$$

where $d\chi^2 = [(1 - kr^2)^{-1}dr^2 + r^2 d\Omega^2]$ is new shorthand for the spatial part of the metric and ξ and η are arbitrary constants. Since none of the metric coefficients depend on ψ, the equation of state is that of radiation:

$$\rho = \frac{3(\xi^2 + 4k\eta)}{32\pi(-kt^2 + \xi t + \eta)^2} \quad , \quad p = \frac{\rho}{3} \quad . \tag{111}$$

This solution was originally discussed by Davidson *et al.* [254]. The second solution reads:

$$d\hat{s}^2 = \frac{(k\psi + \xi/2)^2}{k\psi^2 + \xi\psi + \eta}dt^2 - (k\psi^2 + \xi\psi + \eta)d\chi^2 - d\psi^2 \quad , \tag{112}$$

and has:

$$\rho = \frac{3k}{8\pi(k\psi^2 + \xi\psi + \eta)} \quad , \quad p = -\frac{\rho}{3} \quad . \tag{113}$$

This is the equation of state of "nongravitating matter" discussed in §6.6. The same equation of state characterizes the third solution:

$$d\hat{s}^2 = dt^2 - \frac{1}{4}t^2(e^{\psi} - ke^{-\psi})^2 d\chi^2 - t^2 d\psi^2 \quad , \tag{114}$$

which has:

$$\rho = \frac{3}{8\pi t^2}\frac{1}{(\tanh\psi)^{2k}} \quad . \tag{115}$$

Finally, McManus' fourth solution is given by:

$$d\hat{s}^2 = \psi^2 dt^2 - \frac{1}{4}\psi^2(e^t + ke^{-t})^2 d\chi^2 - d\psi^2 \quad , \tag{116}$$

with:

$$\rho = \frac{3}{8\pi\psi^2} \quad , \quad p = -\rho \quad . \tag{117}$$

This is the equation of state of a vacuum.

Liu & Wesson [173] have extended the search to *non*-separable $k \neq 0$ solutions (109). Instead of eq. (66), they assume metric coefficients of the form:

$$e^{\nu} \equiv L^2(t - \lambda\psi) \quad , \quad e^{\omega} \equiv M^2(t - \lambda\psi) \quad , \quad e^{\mu} \equiv N^2(t - \lambda\psi) \quad , \tag{118}$$

where L, M and N are *wavelike* functions of the argument $(t - \lambda\psi)$, with λ acting as a "wave number." Their solutions turn out to be determined by two relations:

$$\dot{L}^2 + kM^2 = \zeta^2 M^2 \left(\lambda^2 M^2 \frac{\kappa}{L^2} \right) \quad ,$$
$$N^2 = \lambda^2 M^2 - \frac{k}{\zeta^2} + \frac{\kappa}{L^2} \quad , \tag{119}$$

where L (or M) must be supplied, and ζ and κ are integration constants. With L, M and N specified in this way, the induced-matter energy-momentum tensor can be calculated, and the density and pressure of the cosmological induced matter found. If one supposes, for example, that $\kappa = 0$ and $M = L^{(3\gamma+1)/2}$, with γ a new constant, then one obtains a perfect fluid with:

$$\rho = \frac{3\zeta^2\lambda^2}{8\pi L^{3+3\gamma}} \quad , \quad p = \gamma\rho \quad . \tag{120}$$

In this case the metric (109) reads:

$$d\hat{s}^2 = \frac{1}{L^{1+3\gamma}} dt^2 - L^2\chi^2 - \left(\frac{\lambda^2}{L^{1+3\gamma}} - \frac{k}{\zeta^2} \right) d\psi^2 \quad , \tag{121}$$

where $L(t - \lambda\psi)$ plays the role of the *cosmological scale factor*, obeying the field equation:

$$\dot{L}^2 + \frac{k}{L^{1+3\gamma}} = \frac{\zeta^2\lambda^2}{L^{2+6\gamma}} \quad . \tag{122}$$

This solution can be used to describe, for example, the matter-dominated era ($\gamma = 0$) or the radiation-dominated one ($\gamma = 1/3$). The properties of this model are discussed in more detail in [173]. The fact that the scale factor depends on ψ as well as t is particularly interesting, and implies that observers with different values of ψ would disagree on the time elapsed since the big bang; that is, *on the age of the universe*. Rather than being a single event, in fact, the big bang in this picture resembles a sort of shock wave propagating along the fifth dimension. This effect could in principle allow one to constrain the theory using observational data on the age spread of objects such as globular clusters [315]-[317]. This has yet to be investigated in detail.

7.4 Newton's Law, the Continuity Equation, and Horizon Size

Besides the equation of state, there are two other important laws relating ρ and p in cosmology; and it is natural to ask whether or not they are automatically satisfied by the induced-matter fluid. These are the continuity (or mass conservation) equation, or equivalently the first law of thermodynamics $dE + pdV = 0$ (where E is energy and V is the three-dimensional volume); and the equation of motion (or geodesic equation). The former can be written:

$$\frac{\partial}{\partial T}(\rho R^3) + p\frac{\partial}{\partial T}(R^3) = 0 \quad . \tag{123}$$

The latter is in general quite complicated (see §6.9), and we defer discussion of the noncomoving case to the next section. For matter which is comoving with a uniform fluid, however, only the radial direction is of interest and the equation of motion is just Newton's law:

$$\frac{\partial^2 R}{\partial T^2} = -\frac{M}{R^2} \quad , \quad M \equiv \frac{4}{3}\pi R^3(\rho + 3p) \quad , \tag{124}$$

where we have used gravitational, rather than inertial mass, as pressure can be significant in cosmological problems [314]. In these equations one must be careful to use *proper* time $T = \int e^{\nu/2} dt$ and distance $R = \int e^{\omega/2} dr$ rather than the "raw coordinates" t and r (where $r^2 = x^2 + y^2 + z^2$ is comoving radial distance). For the cosmological metric (68) one has just $T = \psi t$ and $R = t^{1/\alpha} \psi^{1/(1-\alpha)} r$, and it is straightforward to show, using eqs. (70) for ρ and p, that the conservation equation (123) and equation of motion (124) are both satisfied [167]. In fact, the same thing is true for *any* spatially-flat perfect-fluid cosmology induced in this way by five-dimensional geometry [168]. As far as Newton's law and the continuity equation are concerned, then, noncompactified Kaluza-Klein cosmology is *indistinguishable from standard cosmology*. To the extent that these laws depend on the field equations, this is not surprising, since the five-dimensional field equations $\hat{G}_{\alpha\beta} = 0$ contain exactly the same information as the usual four-dimensional ones $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$.

There are, however, effects which depend on the metric but not the field equations, and the noncompactified versions of these will in general show departures from standard cosmology. The size of the particle horizon, for example, can be computed directly from the line element (assuming a null geodesic, $d\hat{s}^2 = 0$). For the above "dust-like" metric (107), it reads:

$$d = t_0^{2/3} \int_0^{t_0} \left[\psi_0^2 - 9t^2 \left(\frac{d\psi}{dt} \right)^2 \right]^{1/2} \frac{dt}{t^{2/3}} \quad . \tag{125}$$

This is just the usual (four-dimensional) expression, plus a term in $(d\psi/dt)$. (This term necessarily acts to *reduce* the size of the particle horizon because the extra dimension of the cosmological metric (68) is spacelike.) Similar results are found for the "radiation" metrics (106) and (104) above [168]. The value of the derivative $(d\psi/dt)$ can be evaluated with the help of the full geodesic equation, to which we turn next.

7.5 The Equation of Motion

The general equation of motion, or geodesic equation (99), is also a metric-based relation and will contain nonstandard terms if the fifth dimension is real. Since the cosmological fluid is neutral, we disregard electromagnetic terms. The spatial components ($\mu = i$, with $i = 1, 2, 3$) of eq. (99) then read:

$$\begin{aligned} \frac{d^2 x^i}{ds^2} + \Gamma^i_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} &= \frac{\varepsilon \mathcal{B}^2}{(1 - \varepsilon \mathcal{B}^2/\phi^2)\phi^3} \left[\nabla^i \phi + \left(\frac{\phi}{\mathcal{B}} \frac{d\mathcal{B}}{ds} - \frac{d\phi}{ds} \right) \frac{dx^i}{ds} \right] \\ &\quad - g^{i\lambda} \frac{\partial g_{\lambda\nu}}{\partial x^4} \frac{dx^\nu}{ds} \frac{dx^4}{dx} \quad , \end{aligned} \tag{126}$$

where $\mathcal{B}$ is as given in eq. (98). We then define a five-velocity $\hat{v}^A \equiv dx^A/d\hat{s}$, which is related to the usual four-velocity $v^\alpha \equiv dx^\alpha/ds$ by $v^\alpha = (d\hat{s}/ds)\hat{v}^\alpha$. Using the cosmological metric (68), one can show that, for objects which are *comoving* with the cosmological fluid ($\hat{v}^i = 0$), all terms on the right-hand side of eq. (126) vanish [172]. Comoving objects, in other words, satisfy the spatial components of the five-dimensional geodesic equation in *exactly the same way as in the standard four-dimensional theory*. This echoes the result obtained above for Newton's law.

For *noncomoving* objects, however, the right-hand side of eq. (126) will in general contain nonzero terms involving the spatial velocities, the extra part of the metric, and derivatives of the metric coefficients with respect to the extra coordinate. From the viewpoint of four-dimensional general relativity such terms would appear as violations of the weak equivalence principle or manifestations of a "fifth force" [188], [214], [226]. To put this in practical terms, eq. (126) tells us that *galaxies with*

large peculiar velocities will not necessarily travel along four-dimensional geodesics. Observations of the peculiar motions of galaxies (and groups and clusters of them) are now becoming available [318]-[323], and in principle these can be used to discriminate between noncompactified Kaluza-Klein theory and ordinary general relativity [172], although this has yet to be investigated in detail. Similar considerations will apply to the dynamics of charged test particles such as cosmic rays, for which the electromagnetic terms in eq. (99) would need to be included.

We turn next to the 0- and 4-components of the geodesic equation (99). With comoving spatial coordinates ($\hat{v}^i = 0$) one finds, using the metric (68):

$$\begin{aligned} \frac{d\hat{v}^0}{d\hat{s}} + \frac{2}{\psi}\hat{v}^0\hat{v}^4 + \frac{\alpha^2}{(1-\alpha)^2}\frac{t}{\psi^2}\hat{v}^4\hat{v}^4 &= 0 \quad , \\ \frac{d\hat{v}^4}{d\hat{s}} + \frac{(1-\alpha)^2}{\alpha^2}\frac{\psi}{t^2}\hat{v}^0\hat{v}^0 + \frac{2}{t}\hat{v}^0\hat{v}^4 &= 0 \quad . \end{aligned} \tag{127}$$

A solution of these must be compatible with the metric itself, which imposes the condition:

$$\psi^2\hat{v}^0\hat{v}^0 - \frac{\alpha^2}{(1-\alpha)^2}t^2\hat{v}^4\hat{v}^4 = 1 \quad . \tag{128}$$

From eqs. (127) and (128) the 0- and 4-components of the five-velocity are found [170] to be:

$$\hat{v}^0 = \mp\frac{\alpha}{\sqrt{2\alpha-1}}\frac{1}{\psi} \quad , \quad \hat{v}^4 = \pm\frac{(1-\alpha)^2}{\alpha\sqrt{2\alpha-1}}\frac{1}{t} \quad . \tag{129}$$

The ratio of these gives us the rate of change with time of the extra coordinate ($d\psi/dt = \hat{v}^4/\hat{v}^0$), and this is easily integrated to yield:

$$\psi(t) = \left(\frac{t_0}{t}\right)^{\mathcal{A}} \quad , \tag{130}$$

where t_0 is an integration constant and $\mathcal{A} \equiv (1-\alpha)^2/\alpha^2$. For $\alpha = 2$ (the radiation-dominated era), $\mathcal{A} = 0.25$; while for $\alpha = 3/2$ (the matter-dominated era), $\mathcal{A} \approx 0.11$. The relative rate of change of the extra coordinate is:

$$\frac{d\psi/dt}{\psi} = -\frac{\mathcal{A}}{t} \quad , \tag{131}$$

and this is comfortably small in either case at late times. (This is also true in the STM interpretation, where ψ is related to rest mass; we return to this in the next section.) The small size of $d\psi/dt$ means that the horizon sizes discussed in the last section will be close to those in standard cosmology. The discrepancies are, however, necessarily *nonzero* if the spatial coordinates are chosen to be comoving.

7.6 Cosmological Implications of General Covariance

In the previous section we worked entirely in coordinates defined by the cosmological metric (68). We are of course free to transform to coordinates in which the spatial components of the five-velocity are *not* comoving. For example, we can switch from t, r, ψ to t', r' and ψ', where:

$$t' = t\psi \quad , \quad r' = t^{1/\alpha}r \quad , \quad \psi' = \mathcal{A}t^{\pm\mathcal{A}}\psi \quad . \tag{132}$$

In terms of these new coordinates, the density and pressure of the cosmological fluid are no longer given by eqs. (70) but by:

$$\rho = \frac{3}{8\pi\alpha^2 t'^2} \quad , \quad p = \left(\frac{2\alpha}{3} - 1\right)\rho \quad . \tag{133}$$

These are identical to the expressions in standard early ($\alpha = 2$) and late ($\alpha = 3/2$) cosmology. Also, since the metric transforms as a tensor, $\hat{g}'^{AB} = (\partial x'^A/\partial x^C)(\partial x'^B/\partial x^D)\hat{g}^{CD}$, we have the result that $\hat{g}'^{00} = (2\alpha - 1)/\alpha^2 =$ constant, which implies that in the new coordinates (132) there is a universal or cosmic time. Similarly, using the vector transformation law $\hat{v}'^A = (\partial x'^A/\partial x^B)\hat{v}^B$, we find that the new components of the five-velocity are:

$$\hat{v}'^0 = \mp\frac{\sqrt{2\alpha - 1}}{\alpha} \quad , \quad \hat{v}'^1 = \mp\frac{1}{\sqrt{2\alpha - 1}}\frac{r'}{t'} \quad , \quad \hat{v}'^2 = \hat{v}'^3 = \hat{v}'^4 = 0 \quad . \tag{134}$$

The 0-component of the test particle velocity (which in four-dimensional theory is related to its energy) is *constant*. The first component is proportional to r', which represents a version of *Hubble's law*. And the fourth component vanishes. Taken together, the above observations tell us that the new coordinates defined by eq. (132) - the ones in which $\hat{v}^4 = d\psi/d\hat{s} = 0$ - are *just the ones which give back standard cosmology*. In a fully covariant five-dimensional theory, there can be no *a priori* reason to prefer coordinates in which $\hat{v}^4 = 0$ over those in which $\hat{v}^4 \neq 0$. It is a matter for experiment to decide. As emphasized in §6.7, the choice between the coordinates defined by the metric (68) and those defined by (132) is *not arbitrary* - not as long as the laws of physics are written in terms of four-dimensional concepts like density, pressure, and comoving four-velocity [23], [170]. To decide whether or not the coordinates of the last section are appropriate to describe the "real world," one must look for the effects associated with them, like the nongeodesic motion of galaxies with large peculiar velocities.

Another promising possibility arises if one can interpret the fifth dimension physically, since eq. (131) shows explicitly how it will change with time. In particular, in the context of STM theory, where ψ is related to particle rest mass, this equation implies a *slow variation in rest mass* with time:

$$\frac{\dot{m}}{m} = -\frac{\mathcal{A}}{t} \quad . \tag{135}$$

Putting $\mathcal{A} \approx 0.11$ for the matter-dominated era and $t \approx 15 \times 10^9$ yr for the present epoch, we obtain a value of $\dot{m}/m = -7 \times 10^{-12} yr^{-1}$. This is marginally consistent with ranging data from the *Viking* space probe to Mars, where errors are reported as $\pm 4 \times 10^{-12} yr^{-1}$, and $\pm 10 \times 10^{-12} yr^{-1}$; and quite consistent with timing data for the binary pulsar 1913+16, where errors are reported as $\pm 11 \times 10^{-12} yr^{-1}$ [170], [197]. If the STM hypothesis is valid, these data tell us that observation is close to settling the question of whether cosmology is using coordinates with $\hat{v}^4 = 0$ or ones with $\hat{v}^4 \neq 0$.

It may seem unusual that physical effects can depend on the reference frame in which one observes them. In fully covariant Kaluza-Klein theory this is a necessary consequence of trying to measure a higher-dimensional universe with four-dimensional tools. Perhaps the most graphic example of this is the big bang itself. As demonstrated in §6.7, the cosmological metric (68) is five-dimensionally *flat*. The universe may therefore be far simpler than previously suspected, in that it may have zero curvature. What then of the big bang singularity, the Hubble expansion, the microwave background, and primordial nucleosynthesis? In noncompactified Kaluza-Klein cosmology, these phenomena, which are all defined in four-dimensional terms, are in a sense recognized as geometrical illusions - artifacts of a choice of coordinates in the higher-dimensional world [169]. Something like this occurs even in four-dimensional general relativity when one works with *comoving spatial coordinates*, in

which galaxies remain forever apart and there is no initial singularity [170]. Relativity is founded on the idea that there should be no preferred coordinate systems; yet in spatially comoving frames there is no big bang. This paradox has no resolution within Einstein's theory, which must consequently be seen as incomplete. In practice, one usually regards the comoving coordinates as useful but "not real." Noncompactified Kaluza-Klein theory gives us a new way to think about these issues in terms of general covariance in higher dimensions.

8 Astrophysics

8.1 Kaluza-Klein Solitons

To model astrophysical phenomena like the Sun or other stars in Kaluza-Klein theory, one must extend the spherically-symmetric Schwarzschild solution of general relativity to higher dimensions. Birkhoff's theorem guarantees that the four-dimensional Schwarzschild metric is both static and *unique* to within its single free parameter (the mass of the central object). This theorem, however, does not hold in higher dimensions, where solutions that are spherically-symmetric (in three or more spatial dimensions) depend in general on a number of parameters (such as electric and scalar charge) besides mass, and can in some cases be time-dependent as well. Unlike four-dimensional stationary solutions, some can also be nonsingular [226], [324], [325]. Such localized solutions of finite energy can legitimately be called "solitons" in the same broad sense used elsewhere in physics [326]. In fact, some workers [180], [182], [185], [191], [192] have found it convenient to apply this term to the *entire class* of higher-dimensional generalizations of the Schwarzschild metric with finite energy (including those which, technically speaking, do contain geometrical singularities). We follow this convention here.

Kaluza-Klein solitons (in this general sense) were noted as early as 1951 by Heckmann, Jordan & Fricke [327], who found several solutions of the five-dimensional vacuum Einstein equations that were stationary and spherically-symmetric in three-space. Kühnel & Schmutzer [328] carried the problem further in 1961, studying for the first time the motion of test particles in the field of the central mass. (Tangherlini [329] used the alleged instability of such "generalized Keplerian orbits" to argue that there were only three spatial dimensions.) This crucial aspect of Kaluza-Klein theory has been re-examined over the years by several other authors [226], [239], [240], [330], and provides one of the most promising ways to constrain it observationally. We will return to it below.

The first *systematic* studies of stationary Kaluza-Klein solutions with spherical symmetry appeared in 1982 with the work of Chodos & Detweiler [228] and Dobiasch & Maison [331]. The former authors obtained a class of five-dimensional solutions characterized by three parameters (mass plus electric and scalar charge) and emphasized the important point that *solitons are generic to Kaluza-Klein theory in the same way that black holes are to ordinary general relativity*. This is what makes them so important in confronting the theory with experiment. The latter authors worked in D dimensions (although the internal space was restricted to be flat) and their solutions accordingly possess four or more parameters. Various aspects of Chodos-Detweiler and Dobiasch-Maison solitons have been studied in [215], [332].

The physical properties of five-dimensional solitons with zero electric charge were first described in detail by Sorkin [225], Gross & Perry [226], and Davidson & Owen [227], whose solutions (given by eq. (77) in the notation of [227]) are characterized by two parameters. Although these latter authors (along with many others) describe their solutions as "black holes," it is important to note that in some cases the objects being considered are naked singularities [332], or have singular event horizons [215], [333]. The term "monopole" is also potentially misleading since more complicated solitons can, for example, take the form of dipoles [226]. For these reasons we prefer to stay with the

broader term "solitons" in this report.

The Chodos-Detweiler metric was generalized by Gibbons & Wiltshire [334] to include extra nondiagonal terms, introducing a fourth parameter (associated with magnetic charge). These authors also considered the thermodynamics of Kaluza-Klein solitons for the first time. Myers & Perry [335] then extended the discussion to D-dimensional solitons with spherical symmetry in $(D-1)$, rather than three spatial dimensions, which allowed them to obtain Kaluza-Klein versions of the Reissner-Nördstrom and Kerr metrics, as well as the Schwarzschild one. The thermodynamical properties of these objects, especially in six and ten dimensions, were examined by Accetta & Gleiser [336]. Myers [337] considered solitons which were *not* asymptotically Minkowskian. And Yoshimura [166] took the bold step of allowing dependence of his solutions (albeit only the $(D-4)$-dimensional part) on extra dimensions. Others have studied the stability of soliton solutions with respect to classical perturbations [338]-[340] and quantum effects [341].

All this work was done in a higher-dimensional *vacuum*; that is, with no explicit higher-dimensional matter. But most compactified Kaluza-Klein theories, as we have seen, operate in curved higher-dimensional spaces and require such matter (or other modifications of the Einstein equations) to ensure proper compactification, among other things. This is just as true for soliton solutions as cosmological ones. *Non-Abelian* solitons have accordingly been constructed by many authors using, for example, the Freund-Rubin fields of $D = 11$ supergravity [342], suitably defined six-dimensional [343] or seven-dimensional matter fields [344], and various D-dimensional scalar fields [345]-[350]. Others have preferred to stay in a higher-dimensional vacuum, opting for higher-derivative corrections to the Einstein equations, including (quadratic) Gauss-Bonnet [351] and cubic [352] curvature terms; or for modifications of the Kaluza-Klein mechanism such as "local compactification" [353].

8.2 Are Solitons Black Holes?

The rest of this report is concerned with solitons of the five-dimensional Gross-Perry-Davidson-Owen-Sorkin (GPDOS) type [225]-[227], with the line element (77) in the notation of Davidson & Owen. Other spherically-symmetric static solutions, like the class found by Billyard & Wesson [186], and those with more than two independent parameters [193], [194], are subjects for future research. Insofar as the metric coefficients of eq. (77) do not depend on the fifth coordinate, the distinction between compactified and noncompactified approaches is not an issue here. It would, however, become crucial in higher-dimensional generalizations of what follows. We will interpret the four-dimensional properties of Kaluza-Klein solitons as induced by the geometry of *empty five-dimensional space* [20] in the manner of §6.3. When $D > 5$ this requires either a noncompactified approach, or modifications to the field equations, as described at the end of the last section.

The first question to address is whether GPDOS solitons in the induced-matter interpretation can rightly be considered black holes. The two classes of object are alike in one important respect: they contain a curvature singularity at the center of ordinary three-space. However: (1) solitons do not have an event horizon (not as understood in ordinary general relativity, at any rate); and (2) they have an *extended* matter distribution, rather than having all their mass compressed into the central singularity. In this section we try to clarify these properties, which make the term "black hole" an inappropriate one in the context of induced-matter Kaluza-Klein theory.

To begin with, it is apparent from the spatial components of the metric (77) that *the center of the 3-geometry is at* $r = 1/a$ *and not* $r = 0$. The surface area of 2-shells varies as $(ar-1)^{1-\epsilon(k-1)}$, and this shrinks to zero at $r = 1/a$, given that $k > 0$ (as required above for positive density), and that the consistency relation (76) holds. The point $r = 0$ is, in fact, not even part of the manifold, which ends at $r = 1/a$. That this spatial center marks the location of a bona fide curvature singularity, and not merely a coordinate one, may be verified by evaluating the appropriate invariant geometric scalars. The square of the five-dimensional Riemann-Christoffel tensor (or Kretschmann

scalar $\hat{K} \equiv \hat{R}_{ABCD}\hat{R}^{ABCD}$), for example, reads in isotropic coordinates [182]:

$$\hat{K} = \frac{192a^{10}r^6}{(a^2r^2-1)^8}\left(\frac{ar-1}{ar+1}\right)^{4\epsilon(k-1)}\Big[1-2\epsilon(k-1)(2+\epsilon^2k)ar +2(3-\epsilon^4k^2)a^2r^2-2\epsilon(k-1)(2+\epsilon^2k)a^3r^3+a^4r^4\Big] \quad , \tag{136}$$

and this is manifestly divergent at $r = 1/a$ (with $k > 0$). (In the Schwarzschild limit this expression simplifies to $\hat{K} = 192a^{10}r^6(ar+1)^{-12}$, which is formally the same as that in four-dimensional Einstein theory. This however has little significance from the Kaluza-Klein point of view since the point $r = -1/a$ is not in the manifold.) The relevant *four*-dimensional curvature invariant is the square of the Ricci tensor, $C \equiv R_{\alpha\beta}R^{\alpha\beta}$, and this comes out as [182]:

$$C = \frac{8\epsilon^2a^{10}r^6}{(a^2r^2-1)^8}\left(\frac{ar-1}{ar+1}\right)^{2\epsilon(k-1)}\Big[3+4\epsilon(3-2k)ar+2(3+6\epsilon^2+4\epsilon^2k^2 -8\epsilon^2k)a^2r^2+12\epsilon a^3r^3+3a^4r^4\Big] \quad , \tag{137}$$

which is also manifestly divergent at the center of the soliton, $r = 1/a$.

For black holes in general relativity, the event horizon is commonly defined in general coordinates as the surface where the norm of the timelike Killing vector vanishes. In our case the Killing vector is just $(1, 0, 0, 0)$ so its norm vanishes where g_{00} does. For the soliton metric (77) this happens at $r \to 1/a$, given that $k > 0$ and $\epsilon > 0$ (we will find below that physicality requires both these conditions). For physical solitons, in other words, the event horizon shrinks to a point at the center of ordinary space. *Kaluza-Klein solitons must therefore be classified as naked singularities*, as noticed previously by several authors [142], [215], [333], [348]. According to the cosmic censorship hypothesis, such objects should not be realized in nature. The relevance of this (essentially four-dimensional) postulate to five-dimensional objects may, however, be debated. In any case we will show below that if they exist, they could be detectable by conventional astrophysical techniques.

What of the soliton's mass distribution? Applying the standard definition [234] and using the soliton metric (77), one finds [182]:

$$M_g(r) = \frac{2\epsilon k}{a}\left(\frac{ar-1}{ar+1}\right)^{\epsilon} \quad . \tag{138}$$

($G = c = 1$ throughout §8 unless otherwise noted.) This is the gravitational (or Tolman-Whittaker) mass of a Kaluza-Klein soliton as a function of (isotropic) radius r. Other commonly-used definitions of mass can be evaluated [182] but do not lend themselves readily to physical interpretation. For positive mass (as measured at infinity) one must have $\epsilon k > 0$. Since positive density requires in addition that $k > 0$, it is apparent that *both k and ϵ* must be positive for realistic solitons. Eq. (138) therefore implies that *the gravitational mass of the soliton goes to zero at the center* - behaviour which differs radically from that exhibited by black holes. Rather than being concentrated into a pointlike singularity, the mass of the soliton is distributed in an *extended* fashion (although the $\sim 1/r^4$-dependence of density noted above means that this distribution is still a sharply peaked one).

The soliton defined by the Schwarzschild limit is, however, special in this regard. If one simply takes the Schwarzschild values $\epsilon = 0$, $\epsilon k = 1$ and puts them directly into eq. (138), one finds that $M_g(r) = 2/a =$ constant for all r. Replacing the parameter a via $M_* \equiv 2/a$ and putting this into

the metric, eq. (77), one recovers on spacetime sections the four-dimensional Schwarzschild solution (78), with "Schwarzschild mass" M_*. Alternatively, however, one might keep ϵ *arbitrarily small* and allow $r \to 1/a$. In this case one finds that $M_g(r) \to 0$ *irrespective* of ϵ. In other words, there is an ambiguity in the limit by which one is supposed to recover the Schwarzschild solution from the soliton metric. The problem is reminiscent of one investigated by Janis, Newman & Winicour [354], [355] and others [347], [356], [357], in which the presence of a scalar field in four-dimensional general relativity led to ambiguity in defining the center of the geometry. In their case perturbation analysis led to a satisfactory resolution of the problem (in which the Schwarzschild "horizon" at $r = 2M_*$ turned out to be a point). Adopting the same approach, Wesson & Ponce de Leon [182] have conducted a numerical study of eq. (138), and this leads unambiguously to the conclusion that in the Schwarzschild limit (as defined by $k \to \infty$ and $\epsilon \to 0$) the mass does go to zero at $r = 1/a$. The picture that emerges from this numerical work is of an extended cloud of matter whose mass distribution becomes more and more compressed *near* its center as the parameters ϵ and k approach their Schwarzschild values. Due to the nature of the geometry, however, the enclosed gravitational mass *at* the center is always zero.

8.3 Extension to the Time-Dependent Case

The results of the last section make it clear that Kaluza-Klein solitons, although they contain singularities at their centers, are not black holes, since they have neither pointlike mass distributions nor event horizons of the conventional type. A third crucial difference between these two classes of objects, which follows from the fact that Birkhoff's theorem does not hold in five dimensions, is that *soliton metrics can be generalized to include time-dependence*. This goes somewhat against the idea of a soliton as a static solution of the field equations. However, it is reasonable to suppose that solitons, if they exist, must have been formed in some astrophysical or cosmological process during which they could not have been entirely static. So it is of physical, as well as mathematical interest to study the extension to time-dependent solutions.

Liu, Wesson & Ponce de Leon [192] have considered the case in which the coefficients ν, ω and μ of the general spherically-symmetric metric (65) depend not only on the radial coordinate r (as in the GPDOS solution (77)), but on t as well. The metric coefficients are still assumed to be *separable* functions, so that eqs. (6.6) are in effect replaced by:

$$e^{\nu} \equiv A^2(r)T^2(t) \quad , \quad e^{\omega} \equiv B^2(r)U^2(t) \quad , \quad e^{\mu} \equiv C^2(r)V^2(t) \quad . \tag{139}$$

The field equations then produce two sets of differential equations, for which four classes of solutions have been identified. We list these here, with brief comments. All the solutions have $T(t) =$ constant. The first class has $U(t) =$ constant as well, along with $A(r) = C(r)$, and looks like:

$$d\hat{s}^2 = A^2(r)dt^2 - B^2(r)d\sigma^2 - A^2(r)V^2(t)d\psi^2 \quad , \tag{140}$$

where $d\sigma^2 = dr^2 + r^2 d\Omega^2$ as usual, and $V(t)$ can have either an oscillating form $V(t) = \cos(\omega t + \varphi)$ or an exponentially varying one $V(t) = \exp(\pm Ht)$ (the parameters φ and H are arbitrary constants). The four-dimensional parts of these solutions are static, and only the extra-dimensional part varies with time. In the case of the decaying exponential solution, the time-dependent soliton tends toward a static one as $t \gg H^{-1}$.

The second class of solutions has $V(t) = U^{-1}(t)$ and $C(r) = A^{-1/2}(r)$, and can be written in the form:

$$d\hat{s}^2 = A^2(r)dt^2 - U^2(t)B^2(r)d\sigma^2 - U^{-2}(t)A^{-1}(r)d\psi^2 \quad , \tag{141}$$

where $U(t)$ satisfies a differential equation exactly analogous to one in standard FRW *cosmology*, and is given by $U(t) = \sqrt{\varphi + Ht - \kappa t^2}$ (with $\kappa = \pm 1, 0$ playing the role of a curvature constant). This is the most interesting of the time-dependent soliton solutions, and has been looked at separately by Wesson, Liu & Lim [191]. The functions $A(r)$ and $B(r)$ can, for instance, be taken to be the same as those of the static soliton, eqs. (75). The parameters ϵ and k obey the consistency relation (76) as before, and here take the values $1/\sqrt{3}$ and 2 respectively. Choosing in addition $\varphi = 1$ and $\kappa = 0$ for convenience, the metric (141) becomes:

$$\begin{aligned} d\hat{s}^2 &= \left(\frac{ar-1}{ar+1}\right)^{4/\sqrt{3}} dt^2 - \left(\frac{a^2r^2-1}{a^2r^2}\right)^2 \left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}} (1+Ht) d\sigma^2 \\ &\quad - \left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}} (1+Ht)^{-1} d\psi^2 \quad . \end{aligned} \tag{142}$$

In the induced-matter interpretation this geometry manifests itself in four dimensions as matter with anisotropic pressure. Using the same technique as in §6.6 (identifying the pressure three-tensor p_i^j and defining $p \equiv p_i^i/3$), one can nevertheless derive a unique equation of state. This turns out to be [191]:

$$\begin{aligned} \rho &= \frac{a^6 r^4}{3\pi(1+Ht)(a^2r^2-1)^4} \left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}} + \frac{3H^2}{32\pi(1+Ht)^2} \left(\frac{ar+1}{ar-1}\right)^{4/\sqrt{3}} \\ p &= \rho/3 \quad . \end{aligned} \tag{143}$$

The matter comprising this time-dependent soliton satisfies the the relativistic equation of state, as expected since the metric coefficients are all independent of ψ. What is interesting about this solution (142) is that *it reduces to the radiation-dominated cosmological metric* (104) in the limit of zero central mass $a \to \infty$ (ie., $M_* \to 0$). So what began as a metric suitable for astrophysical problems may have cosmological applications, perhaps for modelling solitons in the early universe [191].

The third class of solutions found in [192] has $A(r) =$ constant and $U(t) = V(t) = 1 + Ht$, and reads:

$$d\hat{s}^2 = dt^2 - (1+Ht)^2 \left[B^2(r) d\sigma^2 + C^2(r) d\psi^2\right] \quad . \tag{144}$$

For these solitons the fifth dimension is expanding along with the three-dimensional spatial part. The fourth class of solutions, finally, also has $U(t) = 1 + Ht$, but uses $V(t) = U^\ell(t)$ and $C(r) = [A(r)]^{(\ell+2)/(\ell-1)}$, where ℓ is another arbitrary constant. The associated line element looks like:

$$\begin{aligned} d\hat{s}^2 = \quad & A^2(r) dt^2 - (1+Ht)^2 B^2(r) d\sigma^2 \\ & - (1+Ht)^{2\ell} [A(r)]^{2(\ell+2)/(\ell-1)} d\psi^2 \quad . \end{aligned} \tag{145}$$

For these solitons the three-dimensional space expands, but the fifth dimension can either expand, contract, or remain static accordingly as $\ell > 0$, $\ell < 0$, or $\ell = 0$ respectively.

8.4 Solitons as Dark Matter Candidates

Viewed in four dimensions via the induced-matter mechanism, the soliton resembles a hole in the geometry surrounded by a spherically-symmetric ball of ultra-relativistic matter whose density falls

off at large distances as $1/r^4$. If the universe does have more than four dimensions, these objects should be quite common, being generic to Kaluza-Klein gravity in exactly the same way black holes are to general relativity [180], [228]. It is therefore natural to ask whether they could supply the as-yet undetected *dark matter* which according to many estimates makes up more than 90% of the matter in the universe. Other dark matter candidates, like massive neutrinos or axions, primordial black holes, and a finite-energy vacuum, encounter problems with excessive contributions to the extragalactic background light (EBL) and the cosmic microwave background radiation (CMB), among other things [358]. In view of this we consider here the possibility that the so-called "missing mass" consists of solitons.

Adopting the same approach that has led to strong constraints on some of these other dark matter candidates [358], one can begin by attempting to assess the effects of solitons on *background radiation* [185], assuming that the fluid making up the soliton is in fact composed of photons (although there are no *a priori* reasons to rule out, say, ultra-relativistic neutrinos or gravitons). Rather than guessing at their spectral distribution we restrict ourselves to bolometric calculations for the time being. The soliton density ρ_s at large distances comes from eq. (79), and in the same regime eq. (138) gives $M_g \sim 2\epsilon k/a$. Therefore, for solitons of asymptotic mass M_s:

$$\rho_s \sim \frac{M_s^2}{8\pi c^2 k r^4} \quad , \tag{146}$$

where we have restored conventional units. Because this goes as $1/r^4$ while volume (in the uniform case) increases as only r^3, local density will be overwhelmingly due to just one soliton - the *nearest* one - and we do not need to know about the global distribution of these objects in space. The average separation between solitons of mass M_s in terms of their mean density $\bar{\rho}_s$ is $r = (M_s/\bar{\rho}_s)^{1/3}$, and we can use this as the distance to the nearest one. Writing the mean soliton density as a fraction $\Omega_s \equiv \bar{\rho}_s/\rho_{crit}$ of the critical density $\rho_{crit} = 1.88 \times 10^{-26} h^2 kgm^{-3}$ [359] (where h is the usual Hubble parameter in units of 100 km s^{-1} Mpc^{-1}), we then find that the effective local density (146) of solitonic radiation, expressed as a fraction of the CMB density, is:

$$\frac{\rho_s}{\rho_{CMB}} \sim 5.0 \times 10^{-13} h^{8/3} k^{-1} \Omega_s^{4/3} \left(\frac{M_s}{M_\odot}\right)^{2/3} \quad , \tag{147}$$

where $M_\odot$ stands for one solar mass and $\rho_{CMB} = 2.5 \times 10^{-5} h^{-2} \rho_{crit}$ is the equivalent mass density of the CMB at zero redshift [359]. The quantity k is a free parameter, subject only to the consistency relation (76). A particularly convenient choice for illustrative purposes is $k = 1$ (which implies $\epsilon = 1$ also). This class of solutions was discovered independently by Chatterjee [142] and has the special property that gravitational mass $M_g(r)$ at large distances $r \gg 1/a$ is equal to the Schwarzschild mass M_*. If we suppose that individual solitons have galactic mass ($M_s = 5 \times 10^{11} M_\odot$) and that they collectively make up all the dark matter required to close the universe ($\Omega_s = 0.9$), then eq. (147) tells us that distortions in the CMB will be of the order:

$$\frac{\rho_s}{\rho_{CMB}} \sim 1.0 \times 10^{-5} \quad , \tag{148}$$

where we have used $h = 0.7$ for the Hubble parameter. This is precisely the upper limit set by COBE and other experiments on anomalous contributions to the CMB. So we can conclude that solitons, if they are to provide a significant part of the missing mass, are probably less massive than galaxies.

A similar argument can be made on the basis of *tidal effects*. It is known that conventional dark matter candidates such as black holes can be ruled out if they exceed $\sim 10^8 M_\odot$ in mass, since such objects would excessively distort the shapes of nearby galaxies. The same thing would apply to solitons. However, one has to keep in mind that there is no reason for the parameters k and ϵ to

be equal to one for all solitons. They are not universal constants like c or G, but can in principle vary from soliton to soliton. Those with $k < 1$ will have effective gravitational masses below the corresponding Schwarzschild ones, and will consequently be less strongly constrained. A soliton with $k = 0.1$, for example, will have $\epsilon = 1.05$ from eq. (76), and its gravitational mass (138) at large r will be $M_g = 0.105 M_*$ - only one-tenth the conventional value. And in the extreme case $\epsilon k \to 0$, its gravitational mass will vanish altogether. So while these are promising ways to look for Kaluza-Klein solitons, caution must be taken in interpreting the results. Ideally one would like to be able to apply one or more *independent* tests to a given astrophysical system. We therefore devote the rest of this report to outlining the implications of Kaluza-Klein gravity for the classical tests of general relativity, and for related phenomena such as those having to do with the principle of equivalence.

8.5 The Classical Tests

It is convenient to switch from the notation of Davidson & Owen [227] to that of Gross & Perry [226], and to convert from isotropic coordinates to nonisotropic ones (in which $r' = r(1+GM_*/2r)^2$). The soliton metric (77) then takes the form:

$$\begin{aligned} d\hat{s}^2 &= \left(1 - \frac{2M_*}{r'}\right)^{1/\alpha} dt^2 - \frac{(1 - 2M_*/r')^{(\alpha-\beta-1)/\alpha}}{(1 - 2M_*/r')} dr'^2 \\ &\quad - \left(1 - \frac{2M_*}{r'}\right)^{(\alpha-\beta-1)/\alpha} r'^2 d\Omega^2 - \left(1 - \frac{2M_*}{r'}\right)^{\beta/\alpha} d\psi^2 \quad . \end{aligned} \tag{149}$$

where α and β are related to ϵ and k by $\epsilon = -\beta/\alpha$ and $k = -1/\beta$, and where we have replaced the Gross-Perry parameter m by $M_*/2$. Eq. (149) clearly reduces to the familiar Schwarzschild solution on hypersurfaces $\psi =$ constant as $\alpha \to 1$ and $\beta \to 0$. Defining two new parameters $a \equiv 1/\alpha$ and $b \equiv \beta/\alpha$, together with the function $A(r) \equiv 1 - 2M_*/r$, eq. (149) becomes:

$$d\hat{s}^2 = A^a dt^2 - A^{-(a+b)} dr^2 - A^{(1-a-b)} r^2 d\Omega^2 - A^b d\psi^2 \quad , \tag{150}$$

where we have dropped the primes on r for convenience. The consistency relation (76) takes the form:

$$a^2 + ab + b^2 = 1 \quad . \tag{151}$$

We wish to analyze the motion of photons and massive test particles in the field described by the metric (150). The Lagrangian density $\mathcal{L}$ can be obtained from $\mathcal{L}^2 = g_{ik}\dot{x}^i\dot{x}^k$, where the x^i are generalized coordinates and the overdot denotes differentiation with respect to an affine parameter (such as proper time in the case of massive test particles) along the particle's geodesic trajectory. For the metric (150) this gives:

$$\mathcal{L}^2 = A^a \dot{t}^2 - A^{-(a+b)} \dot{r}^2 - A^{1-a-b} r^2 (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) - A^b \dot{\psi}^2 \quad . \tag{152}$$

From symmetry we can assume that $\dot{\theta} = 0$, so $\theta = \pi/2$ without loss of generality. Application of the Euler-Lagrange equations to the Lagrangian (152) immediately produces three constants of the motion:

$$l \equiv A^a \dot{t} \quad , \quad h \equiv A^{(1-a-b)} r^2 \dot{\phi} \quad , \quad k \equiv A^b \dot{\psi} \quad . \tag{153}$$

The third of these quantities, k, is related to the velocity of the test particle along the fifth dimension. The "Schwarzschild limit" of the theory hereafter refers to the values $a = 1$, $b = 0$ and $k = 0$. With eqs. (152) and (153) we are in a position to describe the motion of photons and test bodies in the weak-field approximation (ie., neglecting terms in $(M_*/r)^2$ and higher orders). The procedure is exactly analogous to that in ordinary general relativity [360], and since details have been given elsewhere [148], [187]-[190] we confine ourselves in what follows to summarizing only the main assumptions and conclusions.

8.6 Gravitational Redshift

This test depends only on the coefficients of the metric (150) and, since the latter is static, one can consider emitters and receivers of light signals with fixed spatial coordinates. The ratio of frequencies of the received and emitted signals is simply:

$$\frac{\nu_r}{\nu_e} = \frac{g_{00}(r_e)}{g_{00}(r_r)} \quad , \tag{154}$$

where r_e and r_r are the positions of the emitter and receiver respectively. Using the metric (150) and discarding terms of second and higher orders in M_*/r, one finds [188] that:

$$\frac{\nu_r - \nu_e}{\nu_e} = aM_* \left(\frac{1}{r_r} - \frac{1}{r_e} \right) \quad . \tag{155}$$

From this result it is clear that the gravitational redshift in Kaluza-Klein theory is in perfect agreement with that of four-dimensional general relativity, as long as one defines the gravitational mass M_g of the soliton by $M_g \equiv aM_*$.

8.7 Light Deflection

The light deflection test is more interesting. Noting that $d\hat{s}^2 = 0$ for photons, and substituting the expressions (153) into eq. (150), one finds the following equation of motion:

$$\left(\frac{dr}{d\phi} \right)^2 - (A^{(2-2a-b)} l^2 - A^{(2-a-2b)} k^2) \frac{r^4}{h^2} + Ar^2 = 0 \quad . \tag{156}$$

For weak fields this can be solved [188] to yield a hyperbolic orbit $r(\phi)$ in which the photon approaches the central mass from infinity at $\phi = 0$ and escapes to infinity along $\phi = \pi + \omega$. The total deflection angle ω is given by:

$$\omega = \frac{4M_*}{r_0} + 2M_* p r_0 \quad , \tag{157}$$

where $p \equiv (2 - a - 2b)(k/h)^2 - (2 - 2a - b)(l/h)^2$ and r_0 is the impact parameter (distance of closest approach to the central mass). The first term in eq. (157) is the familiar Einstein light deflection angle. The second term represents a correction due to the presence of the fifth dimension, and is in principle measurable. (Note that the apparent linear dependence of this term on r_0 is illusory as p involves the square of the "angular momentum" constant $h \propto r_0$ in its denominator.)

The physical meaning of this result can be clarified by using the metric (150) and the definitions (153) to recast eq. (157) in the form [189]:

$$\omega = \frac{4M_*}{r_0}\left[1 - \left(\frac{f - m(d\psi/dt)^2}{1 - n(d\psi/dt)^2}\right)\right] \quad , \tag{158}$$

where:

$$\begin{aligned} f &\equiv (1 - a - b/2)A^{-(1-2a-b)} \quad , \\ m &\equiv (1 - a/2 - b)A^{-(1-3b)} \quad , \quad n \equiv A^{-(a-b)} \quad . \end{aligned} \tag{159}$$

The m- and n-terms can be ignored when the velocity $d\psi/dt$ of the test body along the fifth dimension is negligible. This is certainly true for photons (whose velocity is constant in four dimensions). In addition one can go to the weak-field limit and neglect terms of first order in M_*/r compared to one, so that $A = 1$. In that case $f = 1 - (a + b/2)$ and eq. (158) becomes:

$$\omega = \frac{4M_*}{r_0}\left(a + \frac{b}{2}\right) \quad . \tag{160}$$

This reduces to the general relativistic result in the Schwarzschild limit. For other values of a and b, the Kaluza-Klein light-bending angle will depart from Einstein's prediction, and it is natural to inquire how big such a departure could be. The consistency relation (151) implies that $(a + b/2) = \sqrt{1 - 3b^2/4}$, so in principle eq. (160) is compatible with a range of angles $-\omega_{GR} \leq \omega \leq \omega_{GR}$, where ω_{GR} is the general relativistic value. This would allow for *null deflection* (for $b^2 = 4/3$) and even *light repulsion* (for negative roots). These possibilities are, however, unphysical to the extent that they imply negative values for the (four-dimensional) mass of the soliton. Inertial mass M_i, for example, can be obtained from the Landau pseudo energy-momentum tensor [190], [214], [226], [361], and turns out to be $M_i = (a + b/2)M_*$. Therefore if one requires positivity of inertial mass, then $(a + b/2) \geq 0$, which is incompatible with light repulsion. Similarly, gravitational mass M_g is found from the asymptotic behaviour of $\hat{g}_{00}$ [190], [214], [226], [361] to be given by $M_g = aM_*$ (see also §8.6). (As discussed in these references, and in §8.11, the fact that $M_g \neq M_i$ for $b \neq 0$ need not necessarily constitute a violation of the equivalence principle in Kaluza-Klein theory.) Combining the requirements that $M_g \geq 0$ and $M_i \geq 0$ with the consistency relation (151), one finds that $0 \leq b \leq 1$. Therefore if one requires positivity of both inertial *and* gravitational mass, then the Kaluza-Klein light deflection angle (160) must lie in the range $0.5\omega_{GR} \leq \omega \leq \omega_{GR}$, which rules out null deflection as well as light repulsion.

This however still leaves room for significant departures from general relativity. Why have these not been observed? Most tests to date have been carried out in the solar system which, considered as a soliton, is very close to the limiting Schwarzschild case since nearly all its mass is concentrated near the center. From this perspective the fact that long-baseline interferometric measurements of solar light-bending [197] have confirmed Einstein's prediction to within a factor of $\pm 10^{-3}$ merely tell us - via eq. (160) - that the Sun must have $b < 0.05$. Larger values of this parameter, and hence larger deviations from the predictions of general relativity, might be looked for in the halos of large elliptical galaxies, or in clusters of galaxies, where mass is more evenly distributed. Much of the dark matter is widely believed to be in these places, and if some or all of it is made up of Kaluza-Klein solitons then one could hope to find evidence of anomalous deflection angles in observations of *gravitational lensing* by elliptical galaxies [362], galaxy clusters [363]-[366], and perhaps in observations of *microlensing* by rich clusters [367].

Just as in four-dimensional general relativity, one can also solve the equation of motion (156) for circular, as well as hyperbolic photon orbits. Putting $\dot{r} = 0$ gives [188]:

$$(1 - 2a - b)r^2 + \frac{A}{M_*}r^3 + (b - a)A^{(1-a-2b)}\frac{k^2 r^4}{h^2} = 0 \quad . \tag{161}$$

For negligible motion along the fifth dimension ($k = 0$) this leads to:

$$r = (1 + 2a + b)M_* \quad . \tag{162}$$

In the Schwarzschild limit this gives back the general relativistic result. For other values of a and b, circular photon orbits can occur at other radii. However, prospects for distinguishing between alternative theories of gravity based on this phenomenon are slim [197], so we do not consider it further.

8.8 Perihelion Advance

The elliptical orbits of massive test bodies in orbit around the central mass are of greater interest [187]. Using $d\hat{s}^2 \neq 0$ leads to a slightly more complicated version of the equation of motion (156). This can be solved for the orbit of the test body, which is nearly periodic. The departure from periodicity per orbit, or perihelion shift, is found [188] to be:

$$\delta\phi = \frac{6\pi M_*^2}{h^2}\left(d + \frac{e}{6}\right) \quad , \tag{163}$$

where:

$$\begin{aligned} d &\equiv (1 + k^2) + (a - 1)(-1 + 2l^2 - k^2) + b(-1 + l^2 - 2k^2) \quad , \\ e &\equiv 2(2 - a - b)(-1 + a + b) + 2l^2(-2 + 2a + b)(-1 + 2a + b) \\ &\quad +2k^2(2 - a - 2b)(-1 + a + 2b) \quad . \end{aligned} \tag{164}$$

This gives back the usual general relativistic result in the Schwarzschild limit. If the orbit is nearly circular then eq. (163) can be simplified to read:

$$\delta\phi = \frac{6\pi M_*}{r}\left(a + \frac{2b}{3}\right) \quad , \tag{165}$$

where r is the orbit's coordinate radius. As with the light deflection test, solar system experiments (precession of Mercury's orbit) imply that the Sun, if modelled as a soliton, must have values of a and b very close to the Schwarzschild ones. Extrasolar systems, however, might show nonstandard periastron shifts. Candidate systems could include DI Herculis [368] and AS Camelopardalis [369], as well as binary pulsars [370], x-ray binaries [371], and possibly pulsars with planetary companions [372], [373]. (Eq. (163) would require modifications for systems with significant mass ratios.)

8.9 Time Delay

A similar procedure gives the proper time taken by a photon on a return trip between any two points in the field of the central mass. The definitions (153) and equation of motion (156) lead to the following result [188]:

$$\Delta\tau \;=\; 2\left(1 - \frac{2M_*}{r}\right)^a \left\{ \left(1 + \frac{1}{2}\left(\frac{k}{l}\right)^2\right)\left(\sqrt{r_p^2 - r_o^2} + \sqrt{r_e^2 - r_o^2}\right)\right.$$

$$
\begin{aligned}
&-M_*\left[1+\frac{1}{2}\Big(\frac{k}{l}\Big)^2\right]\left(\frac{\sqrt{r_p^2-r_o^2}}{r_p}+\frac{\sqrt{r_e^2-r_o^2}}{r_e}\right)\\
&+M_*\left[(2a+b)+\frac{3b}{2}\Big(\frac{k}{l}\Big)^2\right]\left[\ln\left(\frac{r_p+\sqrt{r_p^2-r_o^2}}{r_o}\right)\right.\\
&\left.+\ln\left(\frac{r_e+\sqrt{r_e^2-r_o^2}}{r_o}\right)\right]\Bigg\} \quad , \qquad (166)
\end{aligned}
$$

where r_o, r_e and r_p are the photon's distance of closest approach to the central mass and the radius measures to the emitting planet (usually Earth) and reflecting planet respectively, and r is the coordinate radius at which measurement is made (usually the same as r_e). In the Schwarzschild limit, eq. (166) gives back the usual result of four-dimensional general relativity. Experimental data such as that from the Viking spacecraft [197] tell us that our solar system is close to this limit.

8.10 Geodetic Precession

The motion of a *spinning* object in five dimensions is more complicated, but can be usefully studied in at least two important special cases: (1) the case in which the 4 component of the spin vector $\hat{S}^A$ is *zero* [188]; and (2) the case in which the 4-component of spacetime is *flat* [183]. We review these in turn.

The object in both cases is to solve for $\hat{S}^A$ as a function of proper time $\hat{s}$. The requirement of parallel transport implies:

$$
\frac{d\hat{S}^M}{d\hat{s}}+\hat{\Gamma}^M_{AB}\hat{S}^A\hat{v}^B=0 \quad , \qquad (167)
$$

where $\hat{v}^A \equiv dx^A/d\hat{s}$ is the five-velocity. Since $\hat{S}^A$ is spacelike whereas $\hat{v}^B$ is timelike, their inner product can be made to vanish:

$$
\hat{g}_{AB}\hat{S}^A\hat{v}^B=0 \quad . \qquad (168)
$$

Eqs. (167) and (168), together with the metric, can be solved for the components of the spin vector $\hat{S}^A$ if some simplifying assumptions are made.

To evaluate case (1) we use the Gross-Perry soliton metric (150), we restrict ourselves to circular orbits $x^A(\hat{s})$, (for which the velocity vector $\hat{v}^A$ can be written $\hat{v}^A=(t,0,0,td\phi/dt,0)$), and assume that $\hat{S}^4=0$. The resulting expressions for the components of $\hat{S}^A$ are lengthy [188] and not particularly illuminating. The important thing about them is that the spatial components $\hat{S}^i$ show a *rotation relative to the radial direction*, with proper angular speed [188]:

$$
\Omega=\left(\frac{[r-(1+a+b)M_*]r^{(a-1)/2}}{(r-2M_*)^{(a+1)/2}}\right)\Omega_0 \quad , \qquad (169)
$$

where $\Omega_0 \equiv d\phi/dt$ is given by:

$$
\begin{aligned}
\Omega_0 \;=\; &\left[\frac{(1-a-b)}{a}\Big(\frac{1-2M_*}{r}\Big)^{1-2a-b}r^2\right.\\
&\left.+\frac{1}{aM_*}\Big(\frac{1-2M_*}{r}\Big)^{2-2a-b}r^3\right]^{1/2} \quad . \qquad (170)
\end{aligned}
$$

A spin vector $\hat{S}^A$ whose initial orientation is along the radial direction will, after one revolution along $x^A(\hat{s})$, undergo a *geodetic precession*:

$$\delta\phi = 2\pi\left[1 - \frac{\sqrt{r-(1+a+b)M_*}\sqrt{r-(1+2a+b)M_*}}{\sqrt{r(r-2M_*)}}\right] \quad . \tag{171}$$

Going to the weak-field limit and using the consistency relation (151), one can reduce this expression to:

$$\delta\phi = \frac{3\pi M_*}{r}\left(a + \frac{2b}{3}\right) \quad , \tag{172}$$

which gives back the usual general relativistic result in the Schwarzschild limit. In general there will be deviations from Einstein's theory which are in principle measurable. One way to detect them would be with orbiting gyroscopes like those aboard the Gravity Probe-B (GP-B) satellite [374], designed to orbit the earth at an altitude of 650 km. Assuming for the sake of illustration the same value of $b = 0.05$ mentioned in §8.7 as the largest one compatible with solar light-bending experiments, one finds from eq. (172) that the geodetic precession in Kaluza-Klein theory would be 1.238 milliarcseconds per revolution, or an angular rate of 6.674 arcsec yr^{-1}. This exceeds the general relativistic prediction (6.625 arcsec yr^{-1}) by 49 milliarcsec yr^{-1} - a difference that would easily be detected by GP-B. In fact this satellite is expected to measure angular rates as small as 0.1 milliarcsec yr^{-1}, which would allow it to probe b-values as small as 10^{-4}.

We turn now to case (2), in which the spin vector $\hat{S}^A$ is *arbitrary* but the fifth dimension of spacetime is flat. Instead of the soliton metric (150) we introduce a simpler five-dimensional line element [183] which is also spherically-symmetric in three-dimensional space:

$$\begin{aligned} d\hat{s}^2 \;=\; & \frac{\psi^2}{L^2}\left[\left(1 - \frac{2M_*}{r} - \frac{r^2}{L^2}\right)dt^2\right. \\ & \left. -\left(1 - \frac{2M_*}{r} - \frac{r^2}{L^2}\right)^{-1} dr^2 - r^2 d\Omega^2\right] - d\psi^2 \quad . \end{aligned} \tag{173}$$

This reduces to the four-dimensional Schwarzschild-*de Sitter* line element on surfaces $\psi =$ constant:

$$ds^2 = \left(1 - \frac{2M_*}{r} - \frac{r^2}{L^2}\right)dt^2 - \left(1 - \frac{2M_*}{r} - \frac{r^2}{L^2}\right)^{-1} dr^2 - r^2 d\Omega^2 \quad , \tag{174}$$

and the constant L (which has units of length) gives rise to an effective four-dimensional cosmological constant $\Lambda = 3/L^2$ [178], [179]. The four-dimensional universe is characterized by induced matter whose density and pressure are found from eqs. (63) to be given by $\rho = 3/\psi^2$ and $p = -\rho$.

Consider first of all an object which is not spinning. Its orbit $x^A(\hat{s})$ is found using the metric relation (173), which satisfies a consistency relation:

$$\hat{g}_{AB}\hat{v}^A\hat{v}^B = 1 \quad , \tag{175}$$

together with the geodesic equation (96). The $A = 5$ component of this latter equation turns out [183] to be:

$$\frac{d^2\psi}{d\hat{s}^2} + \frac{1}{\psi}\left(\frac{d\psi}{d\hat{s}}\right)^2 + \frac{1}{\psi} = 0 \quad . \tag{176}$$

This has the simple solution $\psi^2 = \psi_m^2 - \hat{s}^2$, where ψ_m = constant. Since eqs. (173) and (174) are related by $d\hat{s}^2 = (\psi/L)^2 ds^2 - d\psi^2$, one finds that:

$$\psi = \frac{\psi_m}{\cosh[(s - s_m)/L]} \quad , \tag{177}$$

where s_m is some fiducial value of the four-dimensional proper time at which $\psi = \psi_m$. Physically, the fifth coordinate in this spacetime expands from zero size to a maximum value of ψ_m, and then contracts back to zero. We are living in the period $s > s_m$, when ψ is decreasing (cf. §7.1) on cosmological timescales. (The length L is large if the cosmological constant Λ is small.)

The spatial components of the geodesic equation (96) for this five-dimensional metric turn out to be identical with the usual four-dimensional ones [178], [183]. This is somewhat surprising since the four-dimensional metric (174) depends on ψ. It means that the classical tests of relativity discussed in §8.6 - §8.9 are by themselves insufficient to distinguish between Einstein's theory and its five-dimensional counterpart. *When spin is included*, however, the two theories lead to very different predictions. For this we require the full machinery of eqs. (167) and (168) as well as eqs. (96) and (175). Consider for simplicity a circular orbit as before, and assume that the spin vector lies in the plane of the orbit (so $\hat{S}^2 = 0$), and that one can arrange mechanically to satisfy the inequality $r\hat{S}^3 \ll \hat{S}^1$. (These conditions are close to those in the GP-B experiment, or alternatively might be used to model the Sun-Uranus system, since the spin axis of Uranus lies near its orbital plane.) In this case the four equations noted above allow one to solve for all the components of the spin vector (including $\hat{S}^4$). Its precession from the radial direction after one orbit turns out in the weak-field limit ($r^2/L^2 \ll M_*/r \ll 1$) to be [183]:

$$\delta\phi = \frac{3\pi M_*}{r} - \frac{2\pi H_4 r}{H_1 L \cosh[(s_0 - s_m)/L]} \quad , \tag{178}$$

where s_0 is the value of s at the beginning of the orbit, and H_1 and H_4 are normalized amplitudes of the spin vector along the x^1 and x^4 axes respectively. The first term is the usual geodetic precession of four-dimensional general relativity. The extra term depends on the size H_4/H_1 of the spin component along the fifth dimension, the mass M_* of the central body, and cosmological factors like the elapsed four-dimensional proper time. It also involves *radius* in a manner quite different from that of the first term, which suggests that the two terms could be separated experimentally. Whether this is practical or not has yet to be established, but further investigation is warranted insofar as geodetic precession is the *only* test of relativity which can in principle allow us to distinguish between the five-dimensional metric (173) and the four-dimensional one (174).

8.11 The Equivalence Principle

Many of the above tests show departures from four-dimensional geodesic motion. These could be interpreted as violations of the weak equivalence principle (WEP) by the curvature of the fifth dimension. However, Gross & Perry [226] have argued that they should more appropriately be attributed to a breakdown of *Birkhoff's theorem*, since the underlying theory is fully covariant in five dimensions and involves only gravitational effects. Cho & Park [214] have made similar comments, arguing that the extra dimension acts like a fifth force which is, however, indistinguishable from gravity for an uncharged particle. The nature of the fifth force in noncompactified theory has recently been treated in depth by Mashhoon *et al.* [26]. Here we consider only the simple but dramatic illustration afforded by a test body in radial free fall near a soliton. This is the analog of Galileo's experiments with objects dropped vertically in the Earth's gravitational field. And while this case is somewhat impractical in the context of modern tests of gravity, we will see that it leads to several simple and instructive results.

For vertical free-fall, $d\theta = d\phi = 0$, and the equation of motion (156) leads directly to the following result in terms of the constants (153):

$$\dot{r}^2 = A^b l^2 - A^a k^2 - A^{(a+b)} \quad . \tag{179}$$

For a particle which begins at rest ($\dot{r} = 0$) at $r = r_0$, this equation gives:

$$l^2 = [A(r_0)]^a + [A(r_0)]^{(a-b)} k^2 \quad . \tag{180}$$

Combining eqs. (179) and (180), one obtains the "energy condition" [188]:

$$\begin{aligned} \dot{r}^2 \;=\; & \left[\left(1 - \frac{2M_*}{r}\right)^b \left(1 - \frac{2M_*}{r_0}\right)^{(a-b)} - \left(1 - \frac{2M_*}{r}\right)^a \right] k^2 \\ & + \left(1 - \frac{2M_*}{r}\right)^b \left(1 - \frac{2M_*}{r_0}\right)^a - \left(1 - \frac{2M_*}{r}\right)^{(a+b)} \quad . \end{aligned} \tag{181}$$

In the Schwarzschild limit this gives back the familiar four-dimensional formula $\dot{r}^2 = 2M_*(1/r - 1/r_0)$, which has the same form as the energy equation (for vertical free-fall) in classical Newtonian theory.

The particle's *coordinate velocity* in the r-direction is given by $u_r \equiv dr/dt = \dot{r}ds/dt$, and can be calculated from eq. (181) and the metric (150). It turns out (for $r_0 \to \infty$) to be [188]:

$$\begin{aligned} u_r \;=\; & -\frac{1}{\sqrt{1+k^2}} \left(1 - \frac{2M_*}{r}\right)^a \left\{ \left[\left(1 - \frac{2M_*}{r}\right)^b - \left(1 - \frac{2M_*}{r}\right)^a \right] k^2 \right. \\ & \left. + \left(1 - \frac{2M_*}{r}\right)^b \left[1 - \left(1 - \frac{2M_*}{r}\right)^a \right] \right\}^{1/2} \quad . \end{aligned} \tag{182}$$

This explicitly depends on velocity along the fifth dimension through k. Test particles with nonzero values for this parameter will deviate from geodesic trajectories (in four dimensions) and appear to violate the WEP. The a and b parameters also produce discrepancies with four-dimensional theory. For example, the radius where u_r begins to decrease (as the test particle nears the Schwarzschild surface) differs from the simple value of $r^* = 6M_*$ predicted in Einstein's theory. In the case where $k^2 \ll 1$ one finds instead [188]:

$$r^* = 2M_* \left[1 - \left(\frac{2a+b}{3a+b} \right)^{1/a} \right]^{-1} \quad . \tag{183}$$

This reduces to the general relativistic result in the Schwarzschild limit.

The effects of the fifth dimension can perhaps be most readily appreciated in the particle's *acceleration*, which comes from differentiating eq. (179):

$$\ddot{r} = -\frac{M_*}{r^2} [(a+b) A^{(a+b-1)} - b l^2 A^{(b-1)} + a k^2 A^{(a-1)}] \tag{184}$$

In the Schwarzschild limit ($a = 1, b = 0$) this reduces to:

$$\ddot{r} = -\frac{(1+k^2) M_*}{r^2} \quad , \tag{185}$$

which gives back the familiar four-dimensional result when $k = 0$. In general, though, the particle's hidden velocity in the fifth dimension affects its rate of fall towards the central body in a very significant way. For completeness we note that a particle which has $k = 0$ and starts from rest at infinity (in which case eq. (180) implies $l^2 = 1 + k^2$) will have:

$$\ddot{r} = -\frac{aM_*}{r^2} \quad , \tag{186}$$

at large distances ($r \gg 2M_*$). This confirms that a particle accelerates in the field of the soliton at a rate governed by $M_g = aM_*$ (the gravitational mass) and not M_*. As mentioned in §8.7, neither M_g nor M_* is necessarily the same as the soliton's *inertial* mass M_i in Kaluza-Klein theory, the two quantities being related [190], [214], [226], [361] by:

$$M_i = \left(1 + \frac{b}{2a}\right) M_g \quad . \tag{187}$$

These are strictly identical only in the Schwarzschild limit $b = 0$, and in other cases there will be apparent violations of the WEP. (Note that the factor of two is missing in ref. [226].) Experimentally, one can focus on the quantity:

$$\Delta \equiv \left| \frac{(M_g/M_i)_A - (M_g/M_i)_B}{\frac{1}{2}[(M_g/M_i)_A + (M_g/M_i)_B]} \right| \quad , \tag{188}$$

where the subscripts A and B stand for two objects with different compositions. This is known from experiments on the Earth to be less than about 2×10^{-11}, and would be measured to as little as 10^{-17} by the proposed Satellite Test of the Equivalence Principle (STEP) [375]. If eq. (187) is valid, then one expects two different solitons to have:

$$\Delta \approx \frac{1}{2} \left| \left(\frac{b}{a}\right)_B - \left(\frac{b}{a}\right)_A \right| \quad , \tag{189}$$

which vanishes in the Schwarzschild limit $b = 0$. This relation provides yet another way to probe experimentally for the possible existence of extra dimensions.

9 Conclusions

Kaluza unified Einstein's theory of gravity and Maxwell's theory of electromagnetism by the simple device of letting the indices run over five values instead of four. Other interactions can be included by letting the indices take on even larger values, but in our review we have concentrated on the prototype theory viewed as an extension of general relativity. Klein's contribution was to explain the apparently unobserved nature of the extra dimension by assuming it was rolled up to a small size, and compactified Kaluza-Klein theory remains one of three principal approaches to the subject. Another is to use the extra dimension as an algebraic aid, as in the projective approach. A third version of Kaluza-Klein theory, on which we have spent considerable time since it is the newest, regards the fifth dimension as real but not necessarily a simple length or time. In the space-time-matter theory, it is responsible for mass.

All three versions of Kaluza-Klein theory are viable as judged by experiment and observation. In particular, they cannot be ruled out by the classical tests of relativity or results from astrophysics and cosmology. Indeed, it can be difficult to distinguish between the three main versions of Kaluza-Klein theory at the present time because their observational consequences are often similar. To help

differentiate between its variants and bring the whole subject closer to critical test, we suggest several things: (1) A search for exact solutions of new types. New Kerr-like solutions, for example, would help to model spinning elementary particles. (2) More work on quantization. This is a perennial problem, of course, but the richness of Kaluza-Klein theory may offer new routes to its resolution. (3) An investigation of the physical nature of the fifth dimension. While it is merely a construct in the projective approach, it is real and may become large in certain regimes of exact solutions in the compactified approach. In the noncompactified approach, it is not only real but in principle always observable provided one chooses a coordinate system or gauge that properly brings it out.

We do not wish to prejudge the issue of which if any version of Kaluza-Klein gravity will emerge as superior. However, the progress of physics lies in explaining more phenomena on the basis of theories that are constrained by standards of logic, conciseness and elegance. In this regard, we venture the opinion that the fifth dimension will be needed.

9.1 Acknowledgements and References

One of us (J. O.) would like to acknowledge the early encouragement of three inspirational teachers, A. Klassen, H. Janzen and J. Harder. For comments over the years on higher-dimensional gravity we also thank (in alphabetical order) A. Billyard, B. J. Carr, A. A. Coley, C. W. F. Everitt, T. Fukui, D. Kalligas, G. Lessner, P. Lim, H. Liu, B. Mashhoon, D. J. McManus, J. Ponce de Leon, E. Schmutzer and R. Tavakol. This work was supported financially by the National Science and Engineering Research Council (NSERC) and the National Aeronautics and Space Administration (NASA).

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